Lesson 3

Numerical Integration

Last Week

- Defined the definite integral as limit of Riemann sums.

The *definite integral* of $f(t)$ from $t = a$ to $t = b$.

$$\int_a^b f(t)\,dt = \lim_{n\to\infty}(\text{LHS}) = \lim_{n\to\infty}(\text{RHS})$$

LHS: $$f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t$$

RHS: $$f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t$$

Last Time

- Estimate using left and right hand sums and using area with a grid

If $f(x) \geq 0$, then

$$\int_a^b f(x)\,dx$$

represents the area underneath the curve f between $x = a$ and $x = b$.

Example:

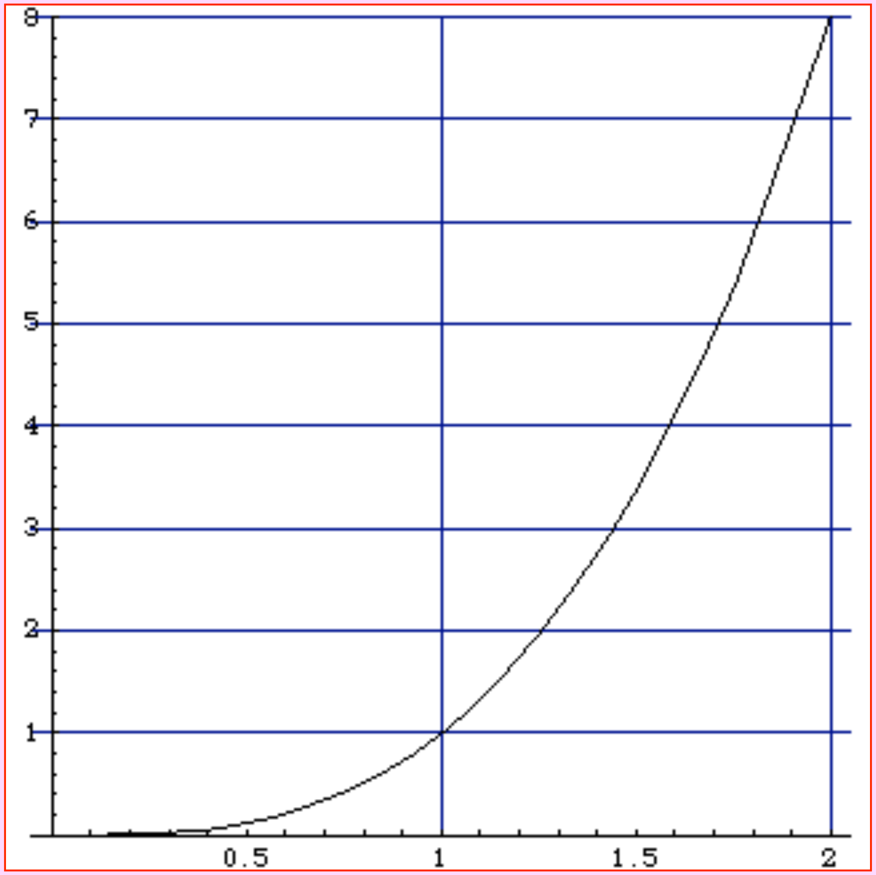
Estimate: $\int_0^2 x^3\,dx$

Note: you have to deal with "partial" boxes.

I estimate about 4 boxes.

Area of each box? 1

So Area = $\int_0^2 x^3\,dx$ =4



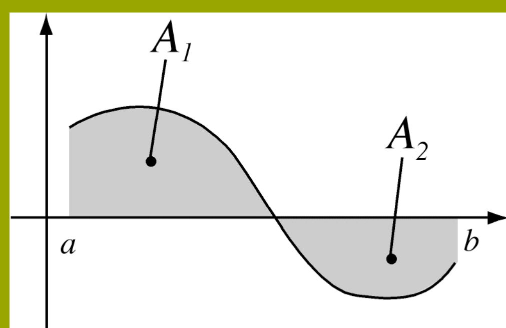
Area Below the Axis

For a general function:

$$\int_a^b f(x)\,dx = (\text{Area above}) - (\text{Area Below})$$

Integral of a rate of change is the total change

Total Change:



$$\int_a^b f(x)\,dx = A_1 - A_2$$

NOTE:
Total Area=

$$A_1 + A_2$$

Group work last time

Find the area under the graph of $y = x^2$ on the interval [1, 3] with $n = 2$ using left rectangles.

$A_L = 1*(1+4) = 5$

Is this estimate an under or over estimate? (Hint: Consider the graph of the function with the rectangles.)

This is an underestimate

Repeat the estimate with right rectangles.

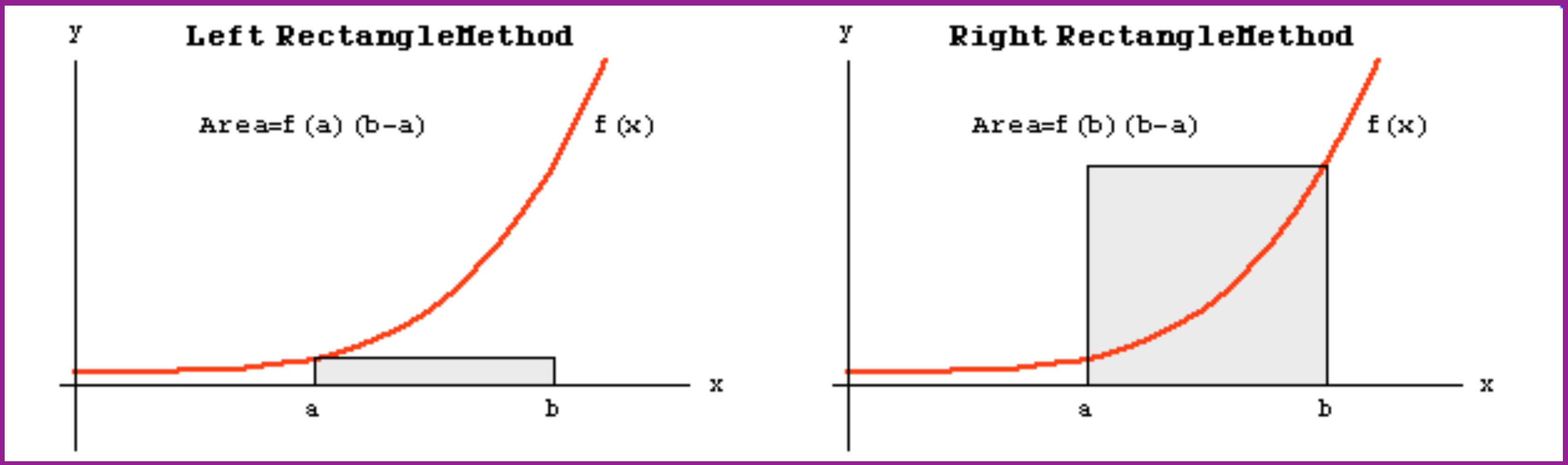
$A_R = 1*(4+9) = 13$, overestimate

Find the average of the two estimates.

$(5+13)/2 = 9$

Estimating Integrals: Trapezoidal and Simpson's Rule

Rectangles(review)



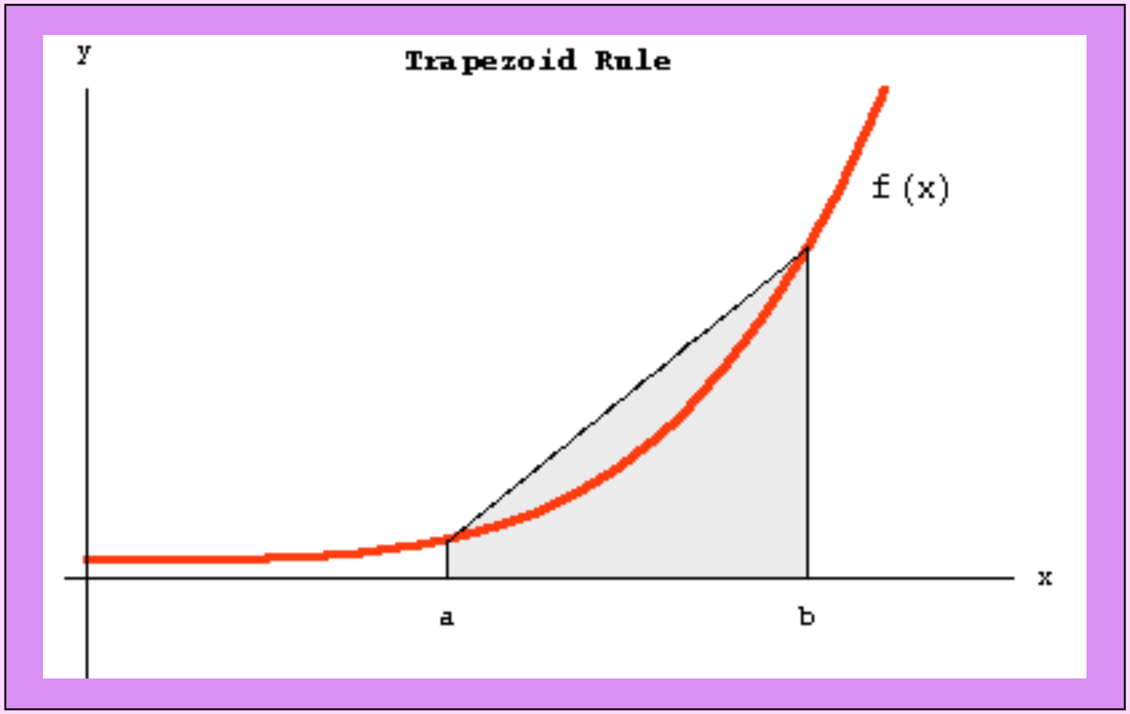
- How can we improve these estimates?

The Trapezoid Rule

- The Trapezoid Rule is simply the average of the left-hand Riemann Sum and the right-hand Riemann Sum.

- Averaging the two Riemann Sums gives an estimate that is more accurate than either sum alone.

A Trapezoid

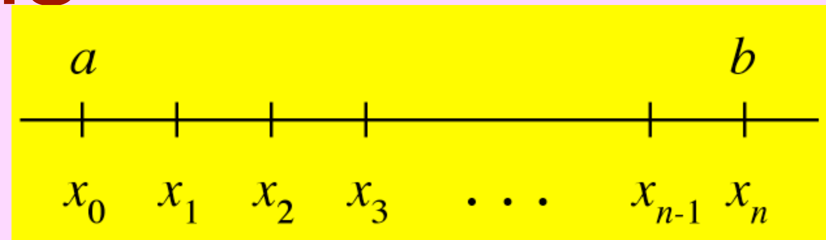


$$\int_a^b f(x)\,dx \approx \frac{[f(a)+f(b)]}{2}(b-a)$$

Notice that the area of the trapezoid is the *average* of the areas of the left and right rectangles

Using Subintervals

Divide the interval into subintervals:



A Formula

Then we get: $\frac{\Delta x}{2}\left[f(x_0)+f(x_1)\right]+\cdots+\frac{\Delta x}{2}\left[f(x_{n-1})+f(x_n)\right]$

Factor out $\Delta x/2$:

$$\frac{\Delta x}{2}\left[\left(f(x_0)+f(x_1)\right)+\left(f(x_1)+f(x_2)\right)+\cdots\left(f(x_{n-1})+f(x_n)\right)\right]$$

Combine duplicate terms:

$$\frac{\Delta x}{2}\left[\left(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right)\right]$$

A Formula: Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx$$

$$\frac{\Delta x}{2}\left[\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)\right]$$

Example

Approximate $\int_0^4 x^2 dx$ using $n = 8$ subintervals.

$\Delta x = (4-0)/8 = 1/2$ $x_0 = 0$ $x_1 = 0.5$ $x_2 = 1$

$$\int_0^4 x^2 dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_n)\right]$$
$$= \frac{1/2}{2}\left[f(0) + 2f(0.5) + 2f(1) + \cdots + f(4)\right]$$
$$= 0.25\left[0 + 2(0.25) + 2(1) + \cdots + 16\right]$$
$$= 21.5$$

Riemann Sums?

Left-Hand Sum:

$$\tfrac{1}{2}\left(0^2+0.5^2+1^2+1.5^2+2^2+2.5^2+3^2+3.5^2\right)=17.5$$

Right-Hand Sum:

$$\tfrac{1}{2}\left(0.5^2+1^2+1.5^2+2^2+2.5^2+3^2+3.5^2+4^2\right)=25.5$$

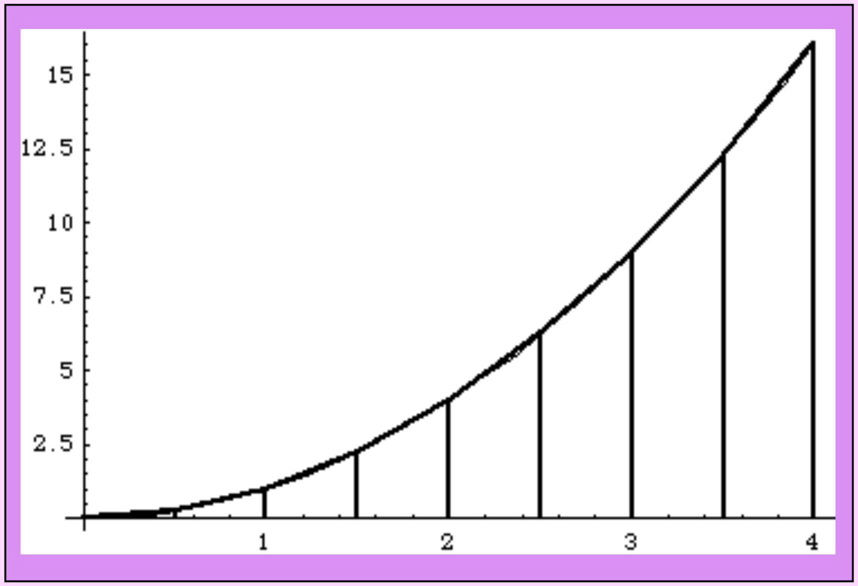
Average: 21.5

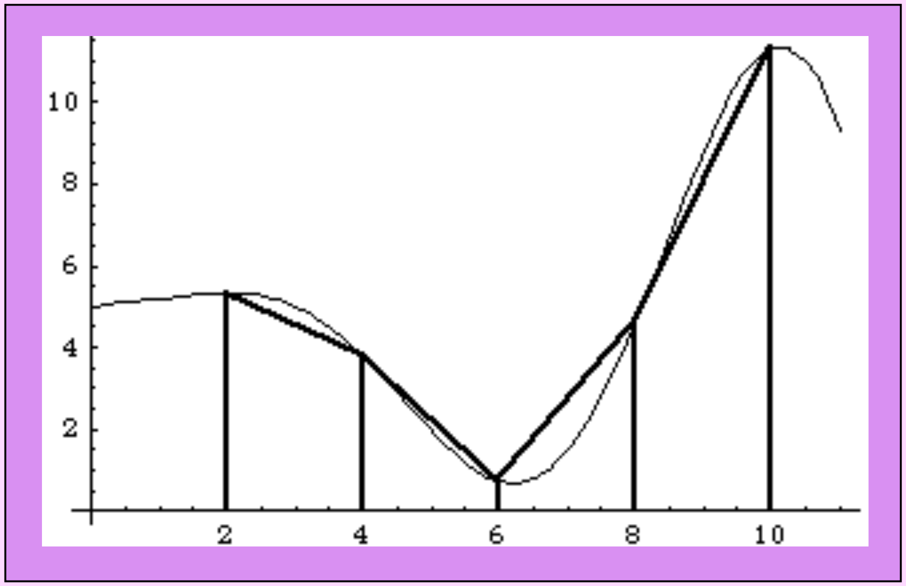
Same as Trapezoidal rule!

Actual answer: $\int_0^4 x^2dx = 64/3 \approx 21.333$

Pictures: Better Approximations

The estimate is pretty good!



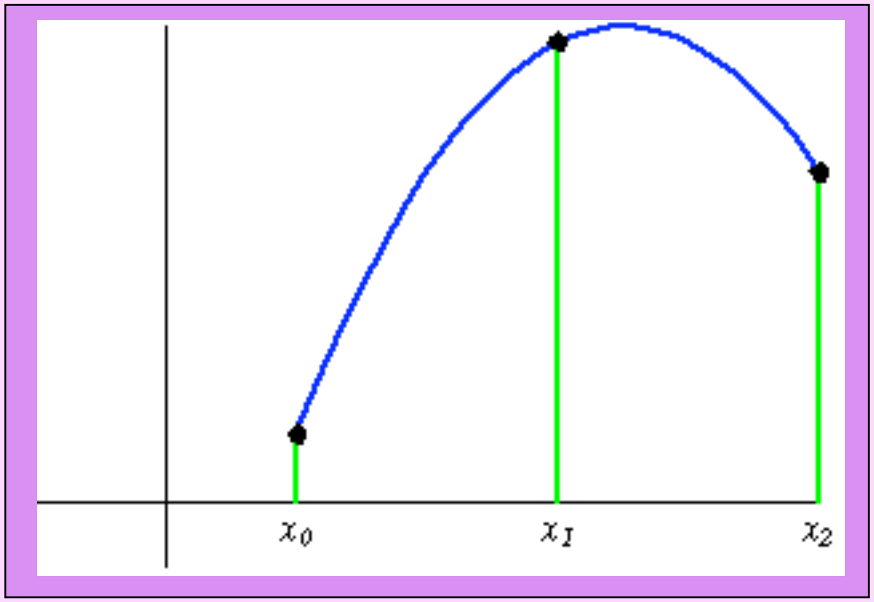


- Trapezoidal uses straight lines: small lines

Next highest degree would be parabolas…

Simpson's Rule

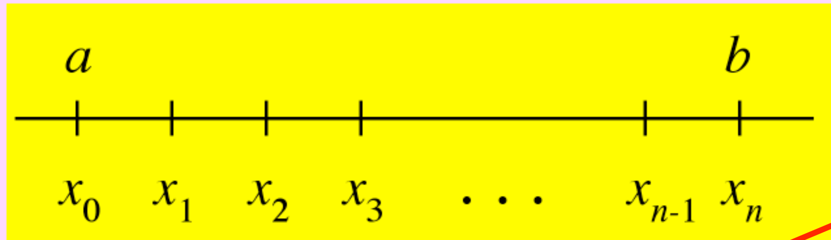
Put a parabola across each pair of subintervals:



So *n* must be *even!*

Simpson's Rule is even more accurate than the Trapezoid Rule.

Simpson's Rule Formula



Second from start and end are both 4

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+4f(x_{n-1})+f(x_n)\right]$$

Like trapezoidal rule

Divide by 3 instead of 2

Interior coefficients alternate: 4,2,4,2,…,4

Example

Estimate $\int_0^4 x^2 dx$ using Simpson's Rule and $n = 4$.

Here, $\Delta x = (4\text{-}0)/4 = 1$.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$
$$= \frac{1}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$
$$= \frac{1}{3}\left[0^2 + 4(1)^2 + 2(2)^2 + 4(3)^2 + (4)^2\right]$$
$$= 64/3 \approx 21.333$$

Exact answer!

Simpson's Rule: Quadratics

Because Simpson's rule uses parabolas,
it is *exact* for any quadratic (or lower) polynomial,
with *any* choice of n.

(So use $n = 2$ for quadratics!)

Tables

- Functions may be represented as tables
- With evenly spaced data, we can still use the Trapezoid and / or Simpson's rule.
- If the number of subintervals is odd, we can only use the Trapezoid rule.

Example:

t	–4	–2	0	2	4
$W(t)$	7	4	3	–1	2

Estimate $\int_{-2}^{4} W(t)\,dt$.

3 subintervals:
use trapezoidal rule.

Here, Δx = ______.

$\Delta x = 2$

$$\int_{-2}^{4} W(t)\,dt \approx \frac{2}{2}\left[W(-2) + 2W(0) + 2W(2) + W(4)\right]$$
$$= \left[4 + 2(3) + 2(-1) + 2\right]$$
$$= 10$$

Example:

Estimate surface area of a pond: Measurements across are taken every 20 feet along the width:

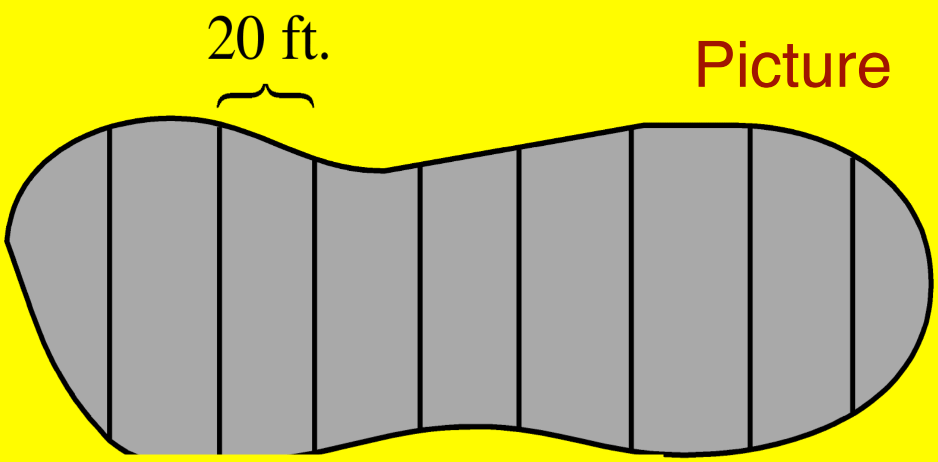
Meas. #	1	2	3	4	5	6	7	8	9
Width (ft)	0	50	54	82	82	73	75	80	0

First: What is Δx? $\Delta x = 20$ ft

Method?

There are 8 subintervals,

so we use Simpson's rule.



Area: $$\frac{20}{3}[0 + 4 \cdot 50 + 2 \cdot 54 + 4 \cdot 82 + 2 \cdot 82$$
$$+ 4 \cdot 73 + 2 \cdot 75 + 4 \cdot 80 + 0] \approx 10{,}413.33 \text{ ft}^2$$

Example: Follow Up

Surface area: 10,413.3 ft^2

If average depth is 10 ft, and we want to start with 1 fish per 1,000 cubic feet of water, how many fish are needed?

(Hint: Start by finding volume.)

Volume: (10,413.3 ft^2)(10 ft) = 104,133 ft^3.

$$\left(\frac{1\text{ fish}}{1{,}000\text{ ft}^3}\right)\left(\frac{104{,}133\text{ ft}^3}{\text{pond}}\right) = 104.133\ \text{fish}/\text{pond}$$

We need about 104 fish.

Review

- The ***Trapezoid Rule*** is nothing more than the average of the left-hand and right-hand Riemann Sums. It provides a more accurate approximation of total change than either sum does alone.

- ***Simpson's Rule*** is a weighted average that results in an even more accurate approximation.

Summary

- Formula for the Trapezoid rule (replaces function with straight line segments)
- Formula for Simpson's rule (uses parabolas, so exact for quadratics)
- Approximations improve as Δx shrinks
- Generally Simpson's rule superior to trapezoidal
- Used both from tabular data

Group work

1. Use Trapezoidal rule and Simpson's rule with 2 subintervals to estimate the following integral:

Trapezoidal rule

$$\int_0^4 x^3 dx$$

$$\approx \frac{2}{2}\left[0^3 + 2(2^3) + 4^3\right]$$

$$= 80.$$

Simpson's rule

$$\int_0^4 x^3 dx$$

$$\approx \frac{2}{3}\left[0^3 + 4(2^3) + 4^3\right]$$

$$= 64.$$

Group work

2. Write down the correct formula to use Simpson's rule and 4 subintervals:

$$\int_2^{10} f(x)dx$$

$$\approx \frac{2}{3}\left[f(2) + 4f(4) + 2f(6) + 4f(8) + f(10) \right]$$