Lesson 13

Solving Definite Integrals
How to find antiderivatives

We have three methods:

1. Basic formulas
2. Algebraic simplification
3. Substitution
## Basic Formulas

<table>
<thead>
<tr>
<th>If ( f(x) ) is…</th>
<th>…then an antiderivative is…</th>
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</thead>
<tbody>
<tr>
<td>( x^n )</td>
<td>( \frac{1}{n+1} x^{n+1} ) except if ( n = -1 )</td>
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</tbody>
</table>
| \( k \)            | \( kx \) (assuming the variable is \( x \)!)
| \( \cos(kx) \)     | \( \sin(kx)/k \) |
| \( \sin(kx) \)     | \( -\cos(kx)/k \) |
| \( e^{kx} \)       | \( e^{kx}/k \) |
| \( \frac{1}{x} \)  | \( \ln|x| \) |
| \( x \)            | \( a^x / \ln(a) \) |
| \( a^x \)          | |

### Notes:
- For \( x^n \), the antiderivative is valid except when \( n = -1 \).
Algebraic Simplification

\[ \int (x - 1)(x + 1) \, dx = \int x^2 - 1 \, dx = \frac{1}{3} x^3 - x + C \]

\[ \int (3x + 1)^2 \, dx = \]

\[ \int 9x^2 + 6x + 1 \, dx = \frac{9}{3} x^3 + \frac{6}{2} x^2 + x + C = 3x^3 + 3x^2 + x + C \]

\[ \int \frac{x + x^2}{x} \, dx = \int \frac{x(1+x)}{x} \, dx = \int 1 + x \, dx = x + \frac{1}{2} x^2 + C \]
Integrals by Substitution

Start with \[ \int f(g(x))g'(x)\,dx \]
Let \( u = g(x) \). \( \frac{du}{dx} = g'(x) \) so \( du = g'(x)\,dx \)
\[ \int f(g(x))g'(x)\,dx = \int f(u)g'(x)\,dx = \int f(u)\,du \]

\[ \int x\sin(x^2)\,dx \]
Let \( u = \) ? \( u = x^2 \) \( \Rightarrow du = 2x\,dx \)
\[ \int x\sin(x^2)\,dx = \frac{1}{2} \int \sin(x^2)2x\,dx = \frac{1}{2} \int \sin(u)\,du = -\frac{1}{2}\cos(x^2) + C \]
Antiderivative Practice

Problem 1 \[ \int 2 \sin(3t) - e^{4t} + 4t \, dt \] Use basic formulas:
\[ \int 2 \sin(3t) - e^{4t} + 4t \, dt = -\frac{2}{3} \cos(3t) - \frac{1}{4} e^{4t} + \frac{1}{\ln(4)} 4^t + C \]

Problem 2 \[ \int \frac{z^2 + 2z}{z^2} \, dz \] Simplify algebraically first, then integrate.
\[ \int \frac{z^2 + 2z}{z^2} \, dz = \int \frac{z^2}{z^2} + \frac{2z}{z^2} \, dz = \int 1 + \frac{2}{z} \, dz = z + 2 \ln |z| + C \]

Problem 3 \[ \int \frac{6t}{4 + t^2} \, dt \] Make a substitution: Let \( u = 4 + t^2 \), so \( du = 2tdt \).
\[ \int \frac{6t}{4 + t^2} \, dt = 3 \int \frac{2t}{4 + t^2} \, dt = 3 \int \frac{1}{u} \, du = 3 \ln |u| + C = 3 \ln |4 + t^2| + C \]
Antiderivative Practice

Problem 4 \[ \int \frac{2}{y \ln(ky)} \, dy \] Make a substitution: \( u=\ln(ky) \), so \( du=dy/y \).

\[ \int \frac{2}{y \ln(ky)} \, dy = \int \frac{2}{\ln(ky)} \frac{1}{y} \, dy = \int \frac{2}{u} \, du = 2 \ln|u| + C = 2 \ln|\ln(ky)| + C \]

Problem 5 Find the particular function \( F(x) \) such that \( F'(x) = x^2 \) and the graph of \( F(x) \) passes through \((1, 2)\).

The general antiderivative is \( \int x^2 \, dx = \frac{1}{3} x^3 + C \)

Then to find \( C \), we must have \( F(1) = \frac{1}{3} 1^3 + C = 2 \)

Thus, \( C = 5/3 \), and our function is \( F(x) = \frac{1}{3} x^3 + \frac{5}{3} \).
Solving Definite Integrals

Theorem: (Fundamental Theorem I) \[ \int_a^b F'(x) \, dx = F(b) - F(a) \]

Or: If \( F \) is an antiderivative for \( f \), then \[ \int_a^b f(x) \, dx = F(b) - F(a) \]

Example

We determined using Simpson’s Rule: \[ \int_0^{12} 3x + 5 \, dx = 276 \]

Now use the fundamental theorem:

An antiderivative for \( f(x) = 3x + 5 \) is \( F(x) = \frac{3}{2} x^2 + 5x \)

So: \[ \int_0^{12} 3x + 5 \, dx = F(12) - F(0) = 276 - 0 = 276 \]
Example

\[ \int_{2}^{3} x^3 \, dx \]

We have to
- find an antiderivative;
- evaluate at 3;
- evaluate at 2;
- subtract the results.

\[ \int_{2}^{3} x^3 \, dx = \left. \frac{1}{4} x^4 \right|_{2}^{3} = \frac{1}{4} 3^4 - \frac{1}{4} 2^4 = \frac{81}{4} - \frac{16}{4} = \frac{65}{4} = 16.25 \]

This notation means: evaluate the function at 3 and 2, and subtract the results.

Don’t need to include “+ C” in our antiderivative, because any antiderivative will work.
Examples

\[ \int_0^\pi 2 \sin(x) + 3x \, dx \]
\[ = -2 \cos x + \frac{3x^2}{2} \bigg|_0^\pi = \left( -2 \cos \pi + \frac{3\pi^2}{2} \right) - \left( -2 \cos 0 + 0 \right) \]

Alternate notation
\[ = \left( -2(-1) + \frac{3\pi^2}{2} \right) - \left( -2(1) + 0 \right) = 4 + \frac{3\pi^2}{2} \]

\[ \int_{-2}^{-1} \frac{1}{s} \, ds \]
\[ = \ln |s| \bigg|_{-2}^{-1} = \ln 1 - \ln 2 = -\ln 2 \]
Practice Examples

\[ \int_1^5 \frac{1}{e} \, dx = \frac{4}{e} \]

\[ \int_2^9 3\sqrt{s} \, ds = \int_2^9 3s^{1/2} \, ds = 2s^{3/2} \bigg|_2^9 = 2(9)^{3/2} - 2(2)^{3/2} = 54 - 4\sqrt{2} \]

\[ \int_{-2}^{-1} \frac{se^s + 1}{s} \, ds = \int_{-2}^{-1} e^s \, ds = (e^s + \ln|s|) \bigg|_{-2}^{-1} = e^{-1} + \ln 1 - (e^{-2} + \ln 2) \]

\[ = \frac{1}{e} - \frac{1}{e^2} - \ln 2 \]
Substitution in Definite Integrals

• We can use substitution in definite integrals.
• However, the limits are in terms of the original variable.
• We get two approaches:
  – Solve an indefinite integral first
  – Change the limits

Method I:

First solve an indefinite integral to find an antiderivative.

Then use that antiderivative to solve the definite integral.

Note: Do not say that a definite and an indefinite integral are equal to each other! They can’t be.
Example

First: Solve an indefinite integral.

\[ \int_{1}^{2} \frac{3t}{t^2 + 4} \, dt \]

\[ u = t^2 + 4 \]
\[ du = 2t \, dt \]

\[ \int \frac{3t}{t^2 + 4} \, dt = \frac{3}{2} \int \frac{2t}{t^2 + 4} \, dt = \frac{3}{2} \int \frac{1}{u} \, du = \frac{3}{2} \ln |t^2 + 4| + C \]

Pull out the 3, and put in a 2.

Second: Use the antiderivative to solve the definite integral.

\[ \int_{1}^{2} \frac{3t}{t^2 + 4} \, dt = \left( \frac{3}{2} \ln |t^2 + 4| \right) \bigg|_{1}^{2} = \left( \frac{3}{2} \ln 8 \right) - \left( \frac{3}{2} \ln 5 \right) = \frac{3}{2} \ln \frac{8}{5} \]

Here’s the antiderivative we just found.
Example

When discussing population growth, we worked backwards to find out what we got from evaluating

\[ \int_a^b \frac{P'(t)}{P(t)} \, dt \]

Let’s find an antiderivative using substitution. \( u = P(t) \)

\[ du = P'(t) \, dt \]

\[ \int \frac{P'(t)}{P(t)} \, dt = \int \frac{1}{u} \, du = \ln|u| + C = \ln|P(t)| + C \]

Of course, \( P(t) \) is always non-negative, so we don’t need absolute values…

So we get:

\[ \int_a^b \frac{P'(t)}{P(t)} \, dt = \ln(P(t)) \bigg|_a^b = \ln(P(b)) - \ln(P(a)) = \ln\left(\frac{P(b)}{P(a)}\right) \]
Method II: Convert the Limits

We start with \( x = a \) and \( x = b \), and a substitution formula \( u = \ldots \)

Just put \( a \) and \( b \) into the substitution formula and get new limits.

Note: You do not have to go back to \( x \) then!

Example \( \int_1^2 \frac{3t}{t^2 + 4} \, dt \) Start with the same substitution

\[
\int_1^2 \frac{3t}{t^2 + 4} \, dt = \frac{3}{2} \int_1^2 \frac{1}{t^2 + 4} \cdot 2t \, dt = \frac{3}{2} \int_5^8 \frac{1}{u} \, du = \left( \frac{3}{2} \ln|u| \right)_5^8 = \frac{3}{2} \ln \frac{8}{5}
\]

What happens to \( t = 1 \)?
\( u = t^2 + 4 = 1^2 + 4 = 5 \).

And when \( t = 2 \),
\( u = t^2 + 4 = 2^2 + 4 = 8 \).
Example

\[ \int_1^8 y \left( \frac{3}{5} y^2 \right) dy \]

\[ u = 5y^2 \]

\[ du = 10y \ dy \]

\[ y = 1: \ u = 5 \]

\[ y = 8: \ u = 320 \]

\[ \int_1^8 y \left( \frac{3}{5} y^2 \right) dy = \int_1^8 y \left( 5y^2 \right)^{\frac{1}{3}} dy = \frac{1}{10} \int_1^8 \left( 5y^2 \right)^{\frac{1}{3}} 10y \ dy = \frac{1}{10} \int_5^{320} \frac{1}{u^3} u^4 \ du = \frac{1}{10} \left[ \frac{u^4}{4} \right]_5^{320} \]

\[ = \frac{3}{40} \left( 320^\frac{4}{3} - 5^\frac{4}{3} \right) \approx 163.5 \]

Note that we can also do this problem without u-sub -- try algebraic simplification

\[ \int_1^8 y \left( \frac{3}{5} y^2 \right) dy = \int_1^8 y \left( 5y^2 \right)^{\frac{1}{3}} dy = \int_1^8 \left( \frac{1}{5^3} y^\frac{2}{3} \right) y \ dy = \frac{1}{5^3} \int_1^8 y^\frac{5}{3} \ dy \]

\[ = \frac{1}{5^3} \left[ \frac{y^{\frac{5}{3} + 1}}{\frac{5}{3} + 1} \right]_1^8 = \frac{1}{5^3} \left( \frac{8^{\frac{5}{3} + 1}}{\frac{5}{3} + 1} \right) = \frac{3}{8} \left( 8^{\frac{8}{3}} - 1^\frac{8}{3} \right) \approx 163.5 \]
Practice Example

\[ \int_0^1 \frac{e^{4x}}{\sqrt{1 + e^{4x}}} \, dx \]

Method I: Firstly compute \( \int \frac{e^{4x}}{\sqrt{1 + e^{4x}}} \, dx \)

\[ u = 1 + e^{4x}, \quad du = 4e^{4x} \, dx \]

\[ \int \frac{e^{4x}}{\sqrt{1 + e^{4x}}} \, dx = \frac{1}{4} \int \frac{4e^{4x}}{\sqrt{1 + e^{4x}}} \, dx = \frac{1}{4} \int \frac{1}{\sqrt{u}} \, du = \frac{1}{4} \int u^{-1/2} \, du = \frac{1}{4} \frac{u^{-1/2 + 1}}{-1/2 + 1} + C \]

\[ = \frac{1}{2} u^{1/2} + C = \frac{\sqrt{1 + e^{4x}}}{2} + C \]

\[ \int_0^1 \frac{e^{4x}}{\sqrt{1 + e^{4x}}} \, dx = \frac{1}{2} \left( \sqrt{1 + e^{4}} \right) \bigg|_0^1 = \frac{1}{2} \sqrt{1 + e^{4}} - \frac{1}{2} \sqrt{1 + e^{0}} = \frac{1}{2} \sqrt{1 + e^{4}} - \frac{1}{2} \sqrt{2} \]

Method II: \( u = 1 + e^{4x} \quad u(0) = 1 + e^{4*0} = 2, u(1) = 1 + e^{4*1} = 1 + e^{4} \)

\[ \int_0^1 \frac{e^{4x}}{\sqrt{1 + e^{4x}}} \, dx = \frac{1}{4} \int_0^1 \frac{4e^{4x}}{\sqrt{1 + e^{4x}}} \, dx = \frac{1}{4} \int_0^1 \frac{1^{1+e^{4}}}{\sqrt{u}} \, du = \frac{1}{4} \int_2^{1+e^{4}} u^{-1/2} \, du = \frac{1}{4} \frac{u^{-1/2 + 1}}{-1/2 + 1} \bigg|_{1+e^{4}}^{2} \]

\[ = \frac{1}{2} u^{1/2} \bigg|_{2}^{1+e^{4}} = \frac{\sqrt{1 + e^{4}^{1+e^{4}}} - \sqrt{2}}{2} \]
Using Definite Integrals

We can now evaluate many of the integrals that we have been able to set up.

Example

Find area between $y = \sin(x)$ and the $x$–axis from $x = 0$ to $x = \pi$, and from $x = 0$ to $x = 2\pi$.

The area from 0 to $\pi$ is clearly: $\int_{0}^{\pi} \sin(x) \, dx = -\cos(x)\bigg|_{0}^{\pi} = 1 + 1 = 2$

The area from 0 to $2\pi$ is more complicated. We note that $\int_{0}^{2\pi} \sin(x) \, dx = 0$

But this is obviously not the area!

The area from 0 to $2\pi$ can be found by:

$\int_{0}^{\pi} \sin(x) \, dx - \int_{\pi}^{2\pi} \sin(x) \, dx = \left(-\cos(x)\bigg|_{0}^{\pi}\right) + \left(\cos(x)\bigg|_{\pi}^{2\pi}\right) = 4$
Summary

• Used the fundamental theorem to evaluate definite integrals.

• Made substitutions in definite integrals
  – By solving an indefinite integral first
  – By changing the limits

• Used the fundamental theorem to evaluate integrals which come from applications.