SECTION 3.3: INDEXED FAMILIES OF SETS

Example 1: For each real number each \( r \in (1, \infty) \) define a set \( A_r \) by

\[
A_r = \{ x \in \mathbb{R} \mid \frac{1}{r} \leq x \leq r \}.
\]

For instance

\[
A_2 = \{ x \in \mathbb{R} \mid \frac{1}{2} \leq x \leq 2 \} = [\frac{1}{2}, 2]
\]

and

\[
A_\pi = \{ x \in \mathbb{R} \mid \frac{1}{\pi} \leq x \leq \pi \} = [\frac{1}{\pi}, \pi].
\]

The collection of all \( A_r \), where \( r \in (1, \infty) \) is an example of an indexed family of sets and we denote such a family of sets by

\[
\{ A_r \}_{r \in (1, \infty)} \quad \text{or by} \quad \{ A_r \mid r \in (1, \infty) \}.
\]

In this example, the set \( (1, \infty) \) is called the indexing set for the family. We can think of \( (1, \infty) \) as a set of labels for our sets.

NOTES

- In the above family of sets there is no set labeled \( A_1 \), there is no “first” set in the family, and given a set \( A_r \) in the family, there is no “next” set in the family. The point is, an infinite collection of sets cannot necessarily be labeled simply as \( A_1, A_2, A_3 \), etc.
- For each set, \( A_r \), in the above family, the index or label \( r \) is associated naturally with the set.
- The indexing set \( (1, \infty) \) contains exactly one label for each set in the family.

Example 2: Recall that \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \). Geometrically, \( \mathbb{R}^2 \) denotes the set of points in the plane. For each point \((a, b) \in \mathbb{R}^2\) define a set \( P_{(a, b)} \) by

\[
P_{(a, b)} = \{ (x, y) \in \mathbb{R}^2 \mid x \geq a \text{ and } y \geq b \}.
\]

For instance,

\[
P_{(1, 2)} = \{ (x, y) \in \mathbb{R}^2 \mid x \geq 1 \text{ and } y \geq 2 \}.
\]

Geometrically, \( P_{(1, 2)} \) consists of those points in the plane to the right of or on the vertical line \( x = 1 \) and above or on the horizontal line \( y = 2 \).

\( \{ P_{(a, b)} \}_{(a, b) \in \mathbb{R}^2} \) is an example of an indexed family of sets in which \( \mathbb{R}^2 \) is the indexing set, or set of labels.
To explore properties of indexed families of sets requires notation for arbitrary indexed families. We will use notation such as \( \{ A_\alpha \}_{\alpha \in \Lambda} \) or \( \{ B_i \}_{i \in I} \) to denote an abstract indexed family of sets.

**Definitions:** Let \( \{ A_\alpha \}_{\alpha \in \Lambda} \) be an indexed family of sets contained in the universal set \( U \). Then

\[
\bigcup_{\alpha \in \Lambda} A_\alpha = \{ x \in U \mid x \in A_\alpha \text{ for some } \alpha \in \Lambda \}.
\]

Similarly,

\[
\bigcap_{\alpha \in \Lambda} A_\alpha = \{ x \in U \mid x \in A_\alpha \text{ for all } \alpha \in \Lambda \}.
\]

**Example 3:** For each real number \( r \in [1, \infty) \) set

\[
A_r = \{ x \in \mathbb{R} \mid -\frac{1}{r} \leq x \leq 2 - \frac{1}{r} \}.
\]

Thus, in interval notation, \( A_r = [-\frac{1}{r}, 2 - \frac{1}{r}] \).

(a) Describe \( \bigcup_{r \in [1, \infty)} A_r \).

(b) Prove that the answer given in (a) is correct.

(c) Describe \( \bigcap_{r \in [1, \infty)} A_r \).

**Solution:** To gain some intuition, we first exhibit some examples of \( A_r \). Thus, \( A_1 = [-1, 1] \), \( A_2 = [-\frac{1}{2}, \frac{3}{2}] \), and \( A_3 = [-\frac{1}{3}, \frac{5}{3}] \).

(a) \( \bigcup_{r \in [1, \infty)} A_r = [-1, 2] \).

(b) To see that \( \bigcup_{r \in [1, \infty)} A_r \subseteq [-1, 2] \), let \( x \in \mathbb{R} \) and suppose \( x \in \bigcup_{r \in [1, \infty)} A_r \). Then there exists \( r \in [1, \infty) \) such that \( x \in A_r \). Then \( x \in [-\frac{1}{r}, 2 - \frac{1}{r}] \); that is, \( -\frac{1}{r} \leq x \leq 2 - \frac{1}{r} \).

Therefore \( -1 \leq -\frac{1}{r} \leq x \leq 2 - \frac{1}{r} < 2 \), so \( x \in [-1, 2] \).

To see that \( [-1, 2] \subseteq \bigcup_{r \in [1, \infty)} A_r \), let \( x \in \mathbb{R} \) and suppose \( x \in [-1, 2) \). We will consider two cases.

**Case 1:** Suppose \( x \in [-1, 1] \). Since \( A_1 = [-1, 1] \), in this case we have \( x \in A_1 \), so \( x \in \bigcup_{r \in [1, \infty)} A_r \).

**Case 2:** Suppose \( x \in (1, 2) \).

**Some Construction:** (not included with the proof) We need to find a real number \( r \geq 1 \) so that \( x \in A_r = [-\frac{1}{r}, 2 - \frac{1}{r}] \). Since \( x > 1 \), let’s focus on finding \( r \) so that \( x \leq 2 - \frac{1}{r} \).

To simplify further, let’s find \( r \) so that \( x = 2 - \frac{1}{r} \). Solving for \( r \) gives \( r = \frac{1}{x-2} \). The proof continues (uninterrupted) as follows.

Set \( r = \frac{1}{x-2} \). Since \( x \in (1, 2) \), \( 2 - x < 1 \), so \( r = \frac{1}{x-2} > 1 \); that is, \( r \in [1, \infty) \). Further, \( 2 - \frac{1}{r} = 2 - (2 - x) = x \), so \( x \in [-\frac{1}{r}, 2 - \frac{1}{r}] = A_r \). Therefore \( x \in \bigcup_{r \in [1, \infty)} A_r \).
This proves that \([-1, 2] \subseteq \bigcup_{r \in [1, \infty)} A_r\), so we conclude that \([-1, 2] = \bigcup_{r \in [1, \infty)} A_r\).

(c) \(\bigcap_{r \in [1, \infty)} A_r = [0, 1]\).

**Exercise 1:** Recall that \(\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}\) is, geometrically, the set of points in the plane. For each real number \(r\), define the subset \(A_r\) of \(\mathbb{R}^2\) by \(A_r = \{(x, y) \in \mathbb{R}^2 \mid y = rx\}\).

(a) Graph \(A_{-1}, A_0,\) and \(A_1\).

(b) Let \(B = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}\) if \(x = 0\) then \(y = 0\).

(Note that \(B\) consists of all points in \(\mathbb{R}^2\) except those on the positive and negative \(y\) axis.)

**Prove** that \(\bigcup_{r \in \mathbb{R}} A_r = B\).

**Hint 1:** To show that \(\bigcup_{r \in \mathbb{R}} A_r \subseteq B\), let \((a, b) \in \mathbb{R}^2\) and suppose \((a, b) \in \bigcup_{r \in \mathbb{R}} A_r\). To prove that \((a, b) \in B\) you need only to show that if \(a = 0\) then \(b = 0\).

**Hint 2:** To show that \(B \subseteq \bigcup_{r \in \mathbb{R}} A_r\), let \((a, b) \in \mathbb{R}^2\) and suppose \((a, b) \in B\). Consider two cases, \(a = 0\) and \(a \neq 0\). In the case where \(a \neq 0\) set \(r = \frac{b}{a}\).

**Theorem 1:** Let \(\{A_\alpha\}_{\alpha \in \Lambda}\) be an indexed family of sets in the universal set \(U\) and let \(B\) be a set in \(U\).

(a) \(B \cup \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B \cup A_\alpha)\).

(b) \(B \cap \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right) = \bigcap_{\alpha \in \Lambda} (B \cap A_\alpha)\).

(c) \(B \cup \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B \cup A_\alpha)\).

(d) \(B \cup \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B \cup A_\alpha)\).

(e) \(\left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)' = \bigcap_{\alpha \in \Lambda} A_\alpha'\).

(f) \(\left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)' = \bigcup_{\alpha \in \Lambda} A_\alpha'\).

**Proof of (c):** To see that \(B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) \subseteq \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)\), let \(x \in U\) and assume that \(x \in B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)\). Then \(x \in B\) and \(x \in \bigcup_{\alpha \in \Lambda} A_\alpha\). Since \(x \in \bigcup_{\alpha \in \Lambda} A_\alpha\), there exists \(\beta \in \Lambda\) such that \(x \in A_\beta\). Therefore, \(x \in B \cap A_\beta\). It follows that \(x \in \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)\). This proves that \(B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) \subseteq \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)\).

To see that \(\bigcup_{\alpha \in \Lambda} (B \cap A_\alpha) \subseteq B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)\) let \(x \in \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)\). Thus, there exists \(\lambda \in \Lambda\) such that \(x \in B \cap A_\lambda\). Therefore, \(x \in B\) and \(x \in A_\lambda\). Since \(x \in A_\lambda\) it follows that \(x \in \bigcup_{\alpha \in \Lambda} A_\alpha\). Consequently, \(x \in B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)\) and it follows that \(\bigcup_{\alpha \in \Lambda} (B \cap A_\alpha) \subseteq B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)\).

We conclude that \(B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)\).

**Exercise 2:** Use Theorem 1 and the basic identities from Section 2.2 to prove that (with notation as in Theorem 1) \(B - \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} (B - A_\alpha)\).
SECTION 3.3 EXERCISES:

3.3.1. For each natural number \( n \) let \( A_n = [0, 3 - \frac{1}{n}] \).
(a) Describe (without proof) \( \bigcup_{n \in \mathbb{N}} A_n \).
(b) Describe (without proof) \( \bigcap_{n \in \mathbb{N}} A_n \).
(c) If \( [0, \infty) \) is the universal set then describe \( \bigcup_{n \in \mathbb{N}} A_n' \) and \( \bigcap_{n \in \mathbb{N}} A_n' \). [cf. Theorem 1 (e) and (f).]

3.3.2. For each positive real number \( \alpha \), define a subset \( A_\alpha \) of the plane by
\[
A_\alpha = \{ (x, y) \mid x^2 + y^2 = \alpha^2 \}.
\]
(a) Give a graphical representation for \( \bigcup \{ A_\alpha \mid \alpha \in [1, 2] \} \).
(b) Set \( B = \{ (x, y) \mid 1 \leq x^2 + y^2 \leq 4 \} \). Prove that \( \bigcup \{ A_\alpha \mid \alpha \in [1, 2] \} = B \). [HINT: To show that \( B \subseteq \bigcup \{ A_\alpha \mid \alpha \in [1, 2] \} \) suppose \( (a, b) \in B \). Now exhibit \( \alpha \) such that \( \alpha \in [1, 2] \) and \( (a, b) \in A_\alpha \).

3.3.3. For each positive real number \( \alpha \), define a subset, \( A_\alpha \), of the plane by
\[
A_\alpha = \{ (x, y) \mid 0 < y \leq -\frac{\alpha}{4} x^2 + \alpha \}.
\]
Thus, \( A_\alpha \) is the set of points in the plane that lie above the \( x \)-axis and on or under the graph of the parabola \( y = -\frac{\alpha}{4} x^2 + \alpha \).
Let \( B \) be the set of points in the plane defined by
\[
B = \{ (x, y) \mid -2 < x < 2 \text{ and } y > 0 \}.
\]
(a) Give a graphical representation of the sets \( A_1 \) and \( A_4 \) and \( B \).
(b) Prove that
\[
\bigcup_{\alpha \in \mathbb{R}^+} A_\alpha = B.
\]
[HINT: To show that \( B \subseteq \bigcup_{\alpha \in \mathbb{R}^+} A_\alpha \), let \( (a, b) \in B \). Show the existence of a positive real number \( \alpha \) such that \( b = -\frac{\alpha}{4} a^2 + \alpha \).]

3.3.4. Let \( \{ A_\alpha \}_{\alpha \in I} \) be a nonempty collection of sets. For any set \( B \) give an elementwise proof that \( B \cup \left( \bigcap_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I}(B \cup A_\alpha) \).

3.3.5. Let \( \{ A_\alpha \}_{\alpha \in I} \) be a nonempty collection of sets. Use basic set equalities (i.e., theorems) to prove that for any set \( B \), \( B - \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I}(B - A_\alpha) \).