Specializations of nonsymmetric Macdonald polynomials at infinity

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Notation

$R \subset \mathbb{R}^n$ reduced irreducible root system

$\alpha_i \in R$ simple roots

$\alpha_i^\vee \in R^\vee$ simple coroots

$s_i$ simple reflections

$W = \langle s_i \rangle$ Weyl group

$\ell(w)$ length function

$w_0$ long element

$Q = \bigoplus \mathbb{Z} \alpha_i$ root lattice

$P$ weight lattice
Notation

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\[ \alpha_i^\vee \in R^\vee \quad \text{simple coroots} \]
\[ s_i \quad \text{simple reflections} \]
\[ W = \langle s_i \rangle \quad \text{Weyl group} \]
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\[ w_0 \quad \text{long element} \]
\[ Q = \bigoplus \mathbb{Z}\alpha_i \quad \text{root lattice} \]
\[ P \quad \text{weight lattice} \]

Untwisted affinization of \( R^\vee \) (results hold more generally)

\[ R^\vee + \mathbb{Z}\delta \quad \text{affine coroots} \]
\[ \alpha_0^\vee = -\theta^\vee + \delta \quad \theta^\vee = \text{highest coroot} \]
\[ W_{\text{aff}} = Q \rtimes W \quad \text{affine Weyl group} \]
\[ W_{\text{ext}} = P \rtimes W \quad \text{extended affine Weyl group} \]
\[ \Pi = W_{\text{ext}}/W_{\text{aff}} \cong P/Q \quad \text{length zero elements} \]
\[ w =: t_{\text{wt}(w)}\text{dir}(w) \quad \text{where } w \in W_{\text{ext}, \text{ wt}(w) \in P, \text{ dir}(w) \in W} \]
The nonsymmetric Macdonald polynomials $E_\lambda(X; q, v)$ lie in the group algebra $\mathbb{Q}(q,v)[P] = \mathbb{Q}(q,v)[X^\lambda : \lambda \in P]$; they form a basis. They are variants of the symmetric Macdonald polynomials $P_\lambda(X; q, v)$, which form a basis of $\mathbb{Q}(q,v)[P]^W$ and generalize the Weyl characters ($q = v^2$), Hall-Littlewood polynomials ($q = 0$), Jack polynomials ($v^2 = q^k$), and other important families of symmetric polynomials.

The $E_\lambda$ can be constructed (and are most naturally defined) using double affine Hecke algebras (Cherednik).
Nonsymmetric Macdonald polynomials

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Demazure-Lusztig operators

\[
T_i = vs_i + \frac{v - v^{-1}}{X^{\alpha_i} - 1}(s_i - 1) \quad \text{where} \quad w(X^\lambda) = X^{w(\lambda)}
\]

These operators are an important ingredient in the construction of the \( E_\lambda \).

\( T_w := T_{i_1} \cdots T_{i_\ell} \) is independent of the reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \).
Example for $R = B_2$

\[ E_{(-1,0)}(X; q, v) = X^{(-1,0)} + \frac{(1 - v)(1 + v)}{1 - qv^2} \left( X^{(0,1)} + X^{(0,-1)} \right) \]

\[ + \frac{(1 - v)(1 + v)(1 - qv^6)}{(1 - qv^2)(1 - qv^3)(1 + qv^3)} X^{(1,0)} \]

\[ + \frac{(1 - v)(1 + v)(1 + qv^2)(1 - qv^4)}{(1 - qv^2)(1 - qv^3)(1 + qv^3)} \]

Remarks

- Sage calculates $E_\lambda(X; q, v)$ for any (affine) type.
- $E_\lambda(X; q, v)$ is well-defined at $q^{\pm 1} = 0$ or $v^{\pm 1} = 0$.
- Let $m_\lambda$ denote the minimal coset representative of $t_\lambda$ for $W_{\text{ext}}/W$. Then $X^\mu$ appears in $E_\lambda(X; q, v)$ iff $m_\mu \leq m_\lambda$ in Bruhat order.
## Some specializations of $E_\lambda(X; q, v)$

<table>
<thead>
<tr>
<th>$q = 0$</th>
<th>$q = \infty$</th>
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<tbody>
<tr>
<td>$p$-adic Iwahori-Spherical functions (Ion)</td>
<td>$p$-adic Iwahori-Whittaker functions (Brubaker-Bump-Licata)</td>
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<tr>
<td>$v = 0$</td>
<td>$v = \infty$</td>
</tr>
<tr>
<td>level-one affine Demazure characters (Ion)</td>
<td>???</td>
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Alcove paths

Let $u, w \in W_{\text{ext}}$ and fix a reduced expression $w = \pi s_{i_1} \cdots s_{i_\ell}$.

**Definition**

An *alcove path* of type $(i_1, \ldots, i_\ell)$ starting at $u$ is a sequence of elements $u_0, u_1, \ldots, u_\ell \in W_{\text{ext}}$ satisfying

$$u_0 = u\pi \quad \text{and} \quad u_k \in \{u_{k-1}, u_{k-1}s_{i_k}\} \quad \text{for} \quad k \geq 1.$$
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By abuse of notation, we write $\mathcal{B}(u, w)$ for the set of alcove paths of type $(i_1, \ldots, i_\ell)$ starting at $u$.

Say that $p$ has a ±-fold at step $k$ if $u_k = u_{k-1}$ and

$$u_{k-1}(\alpha_{i_k}^\vee) \in \mathbb{Z}\delta \pm R_+^\vee.$$
Visualizing alcove paths

Alcoves are connected components of \( \mathbb{R}^n \setminus \bigcup_{\alpha, m} \{ x : \langle \alpha, x \rangle + m = 0 \} \). \( W_{\text{aff}} \) acts simply-transitively on the set of alcoves.

**Examples:** Alcove paths for \( R = B_2 \). \( \alpha_1 = (1, -1), \quad \alpha_2 = (0, 1) \)

\[
\begin{align*}
&u = \text{id}, \quad w = s_1 s_2 s_1 s_0 \\
&\text{no folds}
\end{align*}
\]
\[ u = \text{id}, \quad w = s_1s_2s_1s_0 \]

+-fold at step 1
Ram-Yip formula

Recall that $m_\lambda$ is the minimal coset representative of $t_\lambda$ for $W_{\text{ext}}/W$.

Define $w_\lambda \in W$ by $t_\lambda = m_\lambda w_\lambda$.

Let $\text{wt}(p) = \text{wt}(u_\ell)$, $\text{dir}(p) = \text{dir}(u_\ell)$.

**Theorem (Ram-Yip)**

$$T_u E_\lambda(X; q, v) = v^{-\ell(w_\lambda)} \sum_{p \in B(u, m_\lambda)} X^{\text{wt}(p)} v^{\ell(\text{dir}(p))} f^+(p) f^-(p)$$

Here $f^\pm(p)$ are explicit rational functions of $q, v$ built from the $\pm$-folds.

They are products of terms of the form (where $a, b \geq 0$)

$$\frac{v^{-1} - v}{1 - q^a v^b} \quad \text{for } +$$

$$\frac{(v^{-1} - v) q^a v^b}{1 - q^a v^b} \quad \text{for } -$$
Specialization at $q = \infty$

Let $\mathcal{B}^-(u, w)$ be the set of alcove paths with all folds negative.

Let $|p|$ denote the number of folds in an alcove path $p$.

**Proposition (O.-Shimozono)**

$$E_\lambda(X; \infty, v^{-1}) = v^{\ell(w_0)-2\ell(w_\lambda)} \sum_{p \in \mathcal{B}^-(id,m_\lambda)} X^{\text{wt}(p)} v^{\ell(w_0 \text{dir}(p))} (v^{-1} - v)|p|$$

Schwer proved a similar result at $q = 0$ in terms of positively-folded alcove paths; his result inspired the Ram-Yip formula.
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**Proof.** Use the formula

$$E_\lambda(X^{-1}; q^{-1}, v^{-1}) = v^{\ell(w_0) - 2\ell(w_\lambda)} T_{w_0} E_{-w_0(\lambda)}(X; q, v)$$

and take $q \to 0$ in the Ram-Yip formula for the right-hand side.
Quantum Bruhat graph

Our formula for $v = \infty$ requires the *quantum Bruhat graph*, which has vertices $w \in W$ and directed labeled edges $w \overset{\alpha}{\rightarrow} ws_\alpha$ for $\alpha \in R_+$ and

\[
\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(Bruhat edge)}
\]

or

\[
\ell(ws_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho \rangle + 1 \quad \text{(quantum edge)}
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We also need a projection of $p \in B(u, w)$ to a sequence in $W^{|p|}$, defined by successively deleting simple reflections at fold positions from left to right and taking $\operatorname{dir}$.

Example. Take $u = \text{id}$ and $w = t_{(-1,0)} = s_1s_2s_1s_0$ for $R = B_2$. Let $p$ have folds at steps 1 and 3. Then the projection of $p$ is $(y_0, y_1, y_2)$ where

\[
uw = s_1s_2s_1s_0 \quad y_0 = \operatorname{dir}(s_1s_2s_1s_0) = \text{id}
\]

\[
s_2s_1s_0 \quad y_1 = \operatorname{dir}(s_2s_1s_0) = s_1
\]

\[
s_2 \quad s_0 \quad y_2 = \operatorname{dir}(s_2s_0) = s_2s_1s_2s_1
\]
Specialization at $\nu = \infty$

Let $\overleftarrow{Q\mathcal{B}}(u, w)$ be the subset of $\mathcal{B}(u, w)$ made up of alcove paths that project to reverse paths in the quantum Bruhat graph.

**Theorem (O.-Shimozono)**

$$E_\lambda(X; q^{-1}, \infty) = \sum_{p \in \overleftarrow{Q\mathcal{B}}(\text{id}, m_\lambda)} X^{\text{wt}(p)} q^{n(p)}$$
for explicit $n(p) \in \mathbb{Z}_{\geq 0}$.

**Remarks**

- An analogous result at $\nu = 0$ due to Lenart was our starting point.

- Proof uses the “$T_{w_0}$-formula” but is more subtle than at $q = \infty$.

- Corollary: $E_\lambda(X; q^{-1}, \infty)$ has coefficients in $\mathbb{Z}_{\geq 0}[q]$.

- Cherednik and E. Feigin conjecture a relation to the PBW filtration of level-one affine Demazure modules, for antidominant $\lambda$. 
Example for $R = B_2$

$\lambda = (-1, 0)$ \quad $m_\lambda = t_\lambda = s_1 s_2 s_1 s_0$

Let $p \in B(id, s_1 s_2 s_1 s_0)$ with folds at steps 1 and 3.
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Let $p \in \mathcal{B}(\text{id}, s_1 s_2 s_1 s_0)$ with folds at steps 1 and 3.

Then:

- $\text{wt}(p) = \text{wt}(s_2 s_0) = (1, 0)$
- $p$ projects to the following reverse path in the quantum Bruhat graph

```
\text{id} \xleftarrow{\alpha_1} s_1 \xleftarrow{\alpha_1+2\alpha_2} s_2 s_1 s_2 s_1
```

with both edges quantum.
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$$
E_{(-1,0)}(X; q^{-1}, \infty) = X^{(-1,0)} + q^2 X^{(1,0)} + q \left( X^{(0,-1)} + X^{(0,1)} + X^{(0,0)} \right)
$$