Fibonacci's Forgotten Number

Ezra Brown and Jason C. Brunson

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The Fibonacci numbers 1, 2, 3, 5, 8, 13, 21, 34, ... are familiar to all of us. But there is another number associated with Fibonacci, namely 1;22,07,42,33,04,40. This rational number is an approximation to the real root of the cubic polynomial $x^3 + 2x^2 + 10x - 20$ (see Figure 1), written in *sexagesimal* (base 60) notation. That is,

$$1;22,07,42,33,04,40 = 1 + \frac{22}{60} + \frac{7}{60^2} + \frac{42}{60^3} + \frac{33}{60^4} + \frac{4}{60^5} + \frac{40}{60^6}.$$

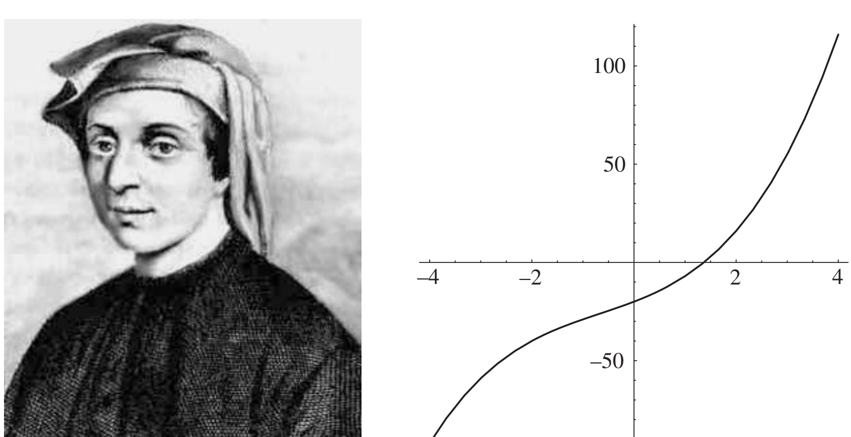

Figure 1. Leonardo Pisano (Fibonacci) and his cubic polynomial $x^3 + 2x^2 + 10x - 20$.

In *decimal* (base 10) this is 1.36880810785, an approximation correct to nine decimal digits (9 DD). Nowadays, computer algebra systems can calculate this in a fraction of a second.

This approximation, however, was found in the thirteenth century, more than 350 years before François Viète's 9 DD approximation to π, 450 years before Newton described the method that bears his name, and over 700 years before the first electronic computer. Who did it? Why was it done? How was it done? ... And why is there a mistake in it? That is our story here.

The *who* is Leonardo of Pisa (ca. 1175–ca. 1250), also known as Fibonacci (of whom no drawings or paintings from his lifetime are known: Figure 1 is a later drawing), arguably the most talented known European mathematician between Pappus of Alexandria (ca. 290–ca. 350) and Regiomontanus (1436–1476). The *why* has to do with a set of problems that were posed to Leonardo as a challenge. The *how*—and the *mistake*—are part of a mystery that we attempt to unravel: the mystery of "Fibonacci's Forgotten Number."

We first describe how Leonardo came to know this number. We then introduce several methods that he may have used for approximating roots of polynomials. Finally, we make a guess as to how he really did it.

Leonardo, *Flos*, and the real root of $x^3 + 2x^2 + 10x = 20$

Leonardo of Pisa was born sometime in the 1170s in Pisa, and died there sometime after 1240. He lived in exciting times. During the Crusades, relations between medieval Europe and the Islamic world were established and flourishing. Pisa was a maritime power with economic connections throughout the Mediterranean world. Europe was rediscovering the great texts from Ancient Greece and Alexandria through the work of Islamic scholars. The scientific and mathematical achievements of India, China, and the Islamic countries were likewise finding their way to European scholars.

In 1192, Leonardo journeyed to Bougia in present-day Algeria to join his father, a customhouse official for the thriving Pisan business community there. While in Bougia he studied under Islamic tutors, learning both the *al-jabr* and the Hindu–Arabic numerals. He travelled the Mediterranean world for about eight years, after which he returned to Pisa and published his monumental *Liber abaci* (*Book of Calculations*) [**12**] in 1202. *Liber abaci* was an introduction to these new ideas and the now-familiar algorithms of arithmetic, as well as their use in a variety of applications familiar to merchants and others engaged in trade. (Chapter 12, Part 7, Problem 18 of *Liber abaci* is the famous rabbits problem whose solution introduces the traditional Fibonacci numbers.) This excellent text cemented Leonardo's reputation as a mathematics teacher and writer, and this reputation spread far and wide. (See [**10**] for a detailed account of Leonardo's life and work; try [**3**] for a somewhat more fanciful account.)

Our story begins in 1225 against this backdrop, at the court of the Holy Roman Emperor Frederick II. Leonardo had been granted a formal audience with the Emperor, a patron of learning who wished to meet Europe's leading mathematician. As a member of this audience, Frederick's court mathematician John of Palermo posed the following problems for Leonardo to solve:

1. Three men possess a pile of money, their shares being $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$. Each man takes some money from the pile until nothing remains. The first returns half of what he took, the second one third and the third one sixth. When the total so returned is divided equally among the men, it is found that each then possesses

what he is entitled to. How much money was in the original pile, and how much did each man take from the pile?

2. Find a rational number r such that both $r^2 - 5$ and $r^2 + 5$ are rational squares.
3. Find a root of the cubic equation $x^3 + 2x^2 + 10x = 20$.

The first problem is fairly routine, and is left for you to solve. Leonardo expanded his solution to the second problem into a treatise he published that same year called *Liber quadratorum* (*Book of Squares*) [**13**]. The challenge appears there as Proposition 17. Leonardo had established that *if* $r^2 - n$ *and* $r^2 + n$ *are rational squares, then there is a right triangle with rational sides, hypotenuse* $2r$*, and area* n. It is not hard to find the relevant triangle for $n = 5$, and hence a solution to the second problem.

We now come to the third problem. According to [**10**], John of Palermo did not create this problem; he just borrowed it from Omar Khayyam's *Al-jabr* (see [**8**]). Leonardo's solution appears in the *Flos*, or *Flower* (see [**6**]), also published in 1225 (certainly a banner year for him), in which he describes all three problems and details his work on the real root of $x^3 + 2x^2 + 10x = 20$, which we hereafter denote μ. He proves that μ is neither an integer, nor rational, nor of any of the forms from Book X of Euclid's *Elements*. He continues ([**10**]), "And because it was not possible to solve this equation in any other of the above ways, I worked to reduce the solution to an approximation." (See also [**6**, Vol. 2, pp. 227–253].) Finally, he states that the root of this equation is approximately 1;22,07,42,33,04,40.

Thus we establish *why*, but several questions immediately come to mind. What methods were available to Leonardo for finding roots of cubics? Which of these (if any) is the most likely method for Leonardo to have used? How did he do it? Why did he give us not the glimmer of an idea about his method? And finally, we note that the sixth base-sixty digit of his approximation is neither a truncation nor a rounding up of the actual root; so, why did he make this mistake?

We begin with what Leonardo knew. By the time he had learned about the Indian figures and the *al-jabr*, the Islamic, Indian, and Chinese mathematicians of the day had developed methods to approximate square roots and cube roots. Unfortunately, these methods would not have helped Leonardo, because the polynomial $x^3 + 2x^2 + 10x - 20$ is not of the form $x^3 = a$ and cannot be placed in that form by "completing the cube." As it happens, there were some other numerical methods with which Leonardo might have had some familiarity. One of them we know these days as the Ruffini–Horner method, so let's talk about that now.

The Ruffini–Horner method

Though named for Paolo Ruffini (1765–1822) and William Horner (1786–1837), two European mathematicians who described it in the early 1800s, the Ruffini–Horner method was first described (as far as we know) by the eleventh-century Chinese mathematician Jia Xian (ca. 1010–ca. 1070), in a now-lost book (see [**5**]). The mathematicians of medieval Islam were very likely familiar with all manner of mathematical texts from both China and India, so it is not unreasonable to suppose that this algorithm was available to Leonardo. Nowadays the algorithm has fallen out of style, so we will describe it in some detail.

Ruffini–Horner finds one digit at a time in the decimal (or sexagesimal) expansion of the desired root by a translation-and-dilation process that resembles integer long division, according to the following basic principles:

Let f be a function and let n be an integer.

1. If $f(x)$ has a root in the interval $[n, n+1)$, then $f(x+n)$ has a corresponding root in $[0, 1)$. We call this *translation by* n.
2. If $p(x) = \sum_{i=0}^{d} a_i x^i$ is a polynomial of degree d with integer coefficients, $10^d p\left(\frac{x}{10}\right) = \sum_{i=0}^{d} a_i 10^{d-i} x^i$ is also a polynomial of degree d with integer coefficients.
3. If $p(x)$ is a polynomial of degree d and has a root in $[0, 1)$, then $10^d p\left(\frac{x}{10}\right)$ has a corresponding root in $[0, 10)$. We call this *dilation by* 10.

Thus, given a polynomial with a root m in $[n, n+1)$, translating by n and dilating by 10 produces a related polynomial with a root in $[0, 10)$.

Now let n_0 be an integer, let $n_1, n_2, \ldots$ be integers in $\{0, 1, \ldots, 9\}$, and let $s = n_0.n_1n_2\ldots$ be the decimal expansion of a root of the polynomial $f(x)$. Invoking the principles above, we see that $s - n_0 = 0.n_1n_2\ldots$ is a root of $f(x+n_0)$ in $[0, 1)$ and $10(r - n_0) = n_1.n_2\ldots$ is a root of $10^d f(\frac{x}{10} + n_0)$ in $[0, 10)$. Thus, if f has a root s and n_0 is the greatest integer of s, then $10^d f(\frac{x}{10} + n_0)$ has a corresponding root whose greatest integer is n_1. Hence, the process of translating and dilating f finds the next digit in the decimal expansion of r—and that is just what Ruffini–Horner does.

There is more: the Ruffini–Horner scheme for polynomial evaluation is more efficient than mere substitution. It begins with the observation that a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ can be rewritten as

$$p(x) = ((\cdots((a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_2)x + a_1)x + a_0.$$

Thus, to evaluate $p(b)$ we multiply a_n by b, add a_{n-1}, multiply by b, add a_{n-2}, and so forth. There is a tabular representation of this method called *synthetic division* that makes all of this easier to see. It looks like this:

$$\begin{array}{c|cccccc} b & a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \\ & & a_n b & a_n b^2 + a_{n-1} b & \ldots & a_n b^n + \cdots + a_1 b \\ \hline & a_n & a_n b + a_{n-1} & a_n b^2 + a_{n-1} b + a_{n-2} & \ldots & a_n b^n + \cdots + a_1 b + a_0 \end{array}$$

Here is how to evaluate $F(3)$, where $F(x) = x^3 + 2x^2 + 10x - 20$ is Fibonacci's cubic:

$$\begin{array}{c|cccc} 3 & 1 & 2 & 10 & -20 \\ & & 3 & 15 & 75 \\ \hline & 1 & 5 & 25 & 55 \end{array}$$

Thus, $F(3) = 55$. For a polynomial of degree d, this method requires d multiplications versus $\binom{d}{2}$ multiplications for the usual substitution method. For that reason, it is the algorithm of choice for doing polynomial evaluation on a computer (see p. 181 of Acton's book [**1**] for further details).

There is even more: along the way, Ruffini–Horner also calculates coefficients for the translated polynomial $p(x+n)$ using iterated synthetic division. Here is why: under the Remainder Theorem, if $p(x)$ is a polynomial with real coefficients and n is a real number, then there is a unique real polynomial $q(x)$ of degree $d-1$ such that

$$q(x) = \frac{p(x) - p(n)}{x - n}.$$

We define real polynomials $p_k(x)$ by $p_0(x) = p(x)$ and $p_{k+1}(x) = \dfrac{p_k(x) - p_k(n)}{x - n}$ for $0 < k < d$. We solve these equations for $p(x) = p_0(x)$ and see that

$$p(x) = p_0(n) + p_1(n)(x-n) + \cdots + p_d(n)(x-n)^d = \sum_{j=0}^{d} p_j(n)(x-n)^j.$$

Thus, we obtain the representation of $p(x)$ in powers of $x - n$ by repeated synthetic division on the polynomials $p(x), p_1(x), p_2(x), \ldots$. We then replace x by $x + n$ to get a representation of the translated polynomial $p(x+n) = \sum_{j=0}^{d} p_j(n)x^j$. Finally, we obtain the dilated polynomial $10^d p(\frac{x}{10} + n)$ by adjoining j zeroes to the coefficient of x^{d-j} in the translated polynomial $p(x+n)$. We observe that this method of calculating the polynomial of degree d translated by n takes $\binom{d}{2}$ multiplications versus $(d+2)(d+1)$ multiplications for the usual method of evaluating $\sum_{i=0}^{d} a_i(x+n)^i$.

Let's watch Ruffini–Horner at work on $F(x) = x^3 + 2x^2 + 10x - 20$. Now F has a unique real root, which we have named μ, and we will derive a decimal representation $\mu = n_0.n_1n_2n_3\ldots$. We find that $F(0) = -20$, $F(1) = -7$, and $F(2) = 16$. Hence $1 < \mu < 2$, and so $n_0 = 1$. We translate F by 1 to obtain $x^3 + 5x^2 + 17x - 7$, and dilate this to obtain $g(x) = x^3 + 50x^2 + 1700x - 7000$. As with long division, we use trial-and-error to find the largest integer n_1 such that $g(n_1) < 0$ and $g(n_1 + 1) > 0$: it turns out that $n_1 = 3$. Let's streamline the process as above:

1	1	2	10	−20	(guess: $n_0 = 1$)
		1	3	13	
	1	3	13	**−7**	
		1	4		
	1	4	**17**		
		1			
	1	**5**			(dilate $x^3 + 5x^2 + 17x - 7$)
3	1	50	1700	−7000	(guess: $n_1 = 3$)
		3	159	5577	
	1	53	1859	**−1423**	
		3	168		
	1	56	**2027**		
		3			
	1	**59**			(dilate $x^3 + 59x^2 + 2027x - 1423$)
6	1	590	202700	−1423000	(guess: $n_2 = 6$)
		6	3576	1237656	
	1	596	206276	**−185344**	
		6	3612		
	1	602	**209888**		
		6			
	1	**608**			(dilate $x^3 + 608x^2 + 209888x - 185344$)

At this point, we see that $\mu = 1.36$ to two digits. If we wantμ to k digits of accuracy, then we translate and dilate k times.

Ruffini–Horner is a clever algorithm—but there was another numerical method available to Leonardo, and he devoted an entire chapter of *Liber abaci* to solving problems by this method, which he called *Elchataym*. We examine it next.

The method of *Elchataym*, or double false position

To dispel any doubts that Leonardo was supremely confident in his problem-solving abilities, we need look no further than Chapter 13 of *Liber abaci*, titled *Here Begins chapter Thirteen on the Method Elchataym and How with It Nearly All Problems of Mathematics Are Solved.* With this bold beginning, Leonardo explains that the Arabic *al-khata'ayn* (literally, "the two errors") is translated as the method of *Double False Position*. Before the development of linear-algebraic techniques six centuries later, this was the standard method for solving linear equations. We now call it "linear interpolation."

Here's how it works. We are looking for the value of x_0 on the line $y = mx + b$ for which $mx_0 + b$ is some given value y_0. To do this, we pick two convenient values x_1 and x_2 and calculate $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$. We know that

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_0 - y_1}{x_0 - x_1}.$$

Solving the latter equality for x_0, we find that

$$x_0 = x_1 + \frac{(y_0 - y_1)(x_2 - x_1)}{y_2 - y_1}.$$

It is clear that this linear interpolation technique can also be used to approximate roots of polynomials. In fact, at the beginning of Chapter 14 of *Liber abaci*, Leonardo uses a modification of this technique to approximate cube roots. Therefore, he could well have used it on the cubic $F(x) = x^3 + 2x^2 + 10x - 20$.

Let's proceed. In *Flos* Leonardo proves, as mentioned above, that our root μ is not an integer: it must lie strictly between 1 and 2 because 0 lies strictly between $F(1)$ and $F(2)$. Let $(x_1, y_1) = (1, -7)$, $(x_2, y_2) = (2, 16)$, and $y_0 = 0$; a short calculation yields the approximation $A = (\frac{30}{23}, 0)$ for the root μ (see Figure 2). (Note that the cubic polynomial $x^3 + 2x^2 + 10x - 20$ is both increasing and concave up, so that the process of linear interpolation always produces an underestimate. We return to this point later.)

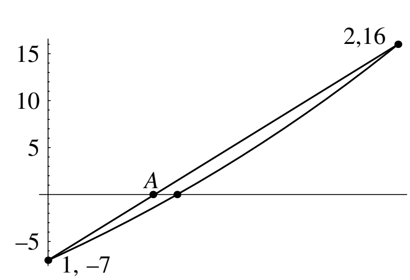

Figure 2. The approximation A of the root μ of $x^3 + 2x^2 + 10x - 20 = 0$ using Elchataym with $x_1 = 1$ and $x_2 = 2$.

Leonardo was probably sharp enough to notice that $3/2$ is also larger than μ. Had he set $x_1 = \frac{30}{23}$ and $x_2 = \frac{3}{2}$, and continued by keeping x_2 fixed and replacing x_1 by the new value of x_0, he would have found the successive values

$$\frac{1068}{781} = 1.367477\ldots,$$
$$\frac{73156892}{53447629} = 1.3687584\ldots, \text{ and}$$
$$\frac{342754819610919692}{250404189091145709} = 1.368806252\ldots.$$

With three more iterations of this algorithm, Leonardo could have obtained the accuracy of the approximation that he attains in the *Flos*.

How did he get his answer?

In this section, we identify what method we think Leonardo used, point out a curious error in his approximation, and speculate on the origins of that error.

As we have seen, Ruffini–Horner and iterated linear interpolation are two numerical methods Leonardo might have used to find that 9 DD approximation to the real root μ of the polynomial $x^3 + 2x^2 + 10x - 20$. There is no record that any other ways to find numerical approximations were available to mathematicians in the early thirteenth century, so we study these two.

We first note that in *Flos*, Leonardo's approximation 1;22,07,42,33,04,40 is a number in base sixty that is slightly greater than μ. This fact may be of use in helping us learn how he arrived at his answer.

Let us now look at Ruffini–Horner. While there is no direct evidence in any of Leonardo's writings that he even knew about the Ruffini–Horner method, it was well-known to the Islamic mathematicians of the day. As Calinger points out (see [**4**, p. 369]), Leonardo could easily have come across it during his travels. Now, the Ruffini–Horner method approximates the root of a polynomial by finding one accurate digit at a time. Like our long division algorithm, each step of the method yields an approximation that is less than or equal to the root in question. and had he used Ruffini–Horner, this approximation would be slightly less than the root μ.

On the other hand, with linear interpolation, we are on firmer ground, because we know that Leonardo used that method in Chapters 13 and 14 of *Liber abaci*. Using linear interpolation on our polynomial also produces approximations that are *under*estimates. As in our example beginning with 1 and 2, iterations of this method *approach from the left*, as the chords connecting points on the curve—e.g., $(1, -7)$ and $(2, 16)$—cross the x-axis to the left of the curve. As mentioned in the previous section, the approximations obtained by beginning with $x_1 = \frac{30}{23}$ and $x_2 = \frac{3}{2}$ are all underestimates of the root μ.

We see that neither of these methods available to Leonardo yield *over*estimates of the root μ. The fact that 1;22,07,42,33,04,40 is slightly more than μ will not help us decide in favor of one method over the other. Because of that, we conclude, based on the prevalence of linear interpolation together with the complete lack of references to Ruffini–Horner in Leonardo's work, that Leonardo applied linear interpolation to John of Palermo's challenge.

But where did the answer 1;22,07,42,33,04,40 come from? The two available methods yield underestimates, but the actual base-sixty expansion of the real root continues 1; 22, 07, 42, 33, 04, 38, 30, 50, ... Thus, it lies just halfway between

1; 22, 07, 42, 33, 04, 38 and 1; 22, 07, 42, 33, 04, 39. The answer he gives is neither truncated nor rounded; why did he give it?

Maybe it was a misprint; misprints happen. Maybe he made a mistake, and even the most expert calculators make mistakes. But maybe he did it deliberately, possibly for the same reason he withheld his method: so that nobody could duplicate his results.

Nowadays, one reason why scholars are eager to publish their work is to establish priority for their results. In Leonardo's day, however, a scholar's methods were stock-in-trade for obtaining well-paying (or, at least, paying) positions with the nobility, the royalty, or the church. As such, scholars kept these methods secret from rivals and competitors. Leonardo's audience with the Emperor was an excellent opportunity for him to display his abilities without revealing his methods. And John of Palermo (having borrowed the cubic polynomial from Khayyam's text, which Leonardo is known to have studied) posed a threat to the secrecy of Leonardo's calculations. This may have been reason enough for Leonardo to further conceal his methods by covering his tracks. Clever Leonardo!

Questions

- *Where can I find out more about this?* Read the Master. Sigler's translations of *Liber abaci* and *Liber quadratorum* ([**12**], [**13**]) are well written and convey both the substance and the flavor of Leonardo's writing. *Flos* is not quite so accessible, but your library may have a copy of Boncompagni's edition of Leonardo's works ([**6**]—*Flos* appears in Vol. 2, pp. 227–253). (*Note:* If you don't know Latin, you may find it tough going. If you do know Latin, you may find it slightly easier . . . and you might even help make it easier for the rest of us by translating it!)
- *Does Khayyam's book describe any methods for solving cubics, and if so, why didn't Leonardo use them?* Yes, it does, but the methods are purely geometric. Omar found the largest positive root r of a cubic equation by a construction involving a semicircle and a rectangular hyperbola. His construction produces a line segment of length μ—but that's another story.
- *How long would it have taken Leonardo to calculate μ to nine digits of accuracy, using Ruffini–Horner or iterated linear interpolation?* This is a fair question. In his time, Leonardo was perhaps the most highly skilled person in the world in paper-and-pencil (or quill-and-parchment) arithmetic. Just as a test, one of us used Ruffini–Horner on Leonardo's problem, beginning with several sheets of blank paper, working in a noisy room, and making three spectacular mistakes in arithmetic along the way. All in all, it took just under two hours. Leonardo would probably have taken less than an hour. A similar experiment using iterated linear interpolation, beginning with $x_1 = \frac{4}{3}$ and $x_2 = \frac{11}{8}$ took between two and three hours. Leonardo could have done it in half that time.
- *Are there any other appearances of the Ruffini–Horner method between Jia Xian and Ruffini?* Partly. According to Rheinboldt (see [**11**]), the Ruffini–Horner method of polynomial evaluation appeared in 1711 in a work by Isaac Newton [**9**], whose own root-finding method dates back to . . . but that too is another story.
- *Are there other works that deal with Leonardo's forgotten number?* In fact, there is at least one other, a paper by Glushkov [**7**] from the 1970s. He does not consider the use of Ruffini–Horner, but he also comes to the conclusion that Leonardo used iterated linear interpolation.
- *Why did Leonardo express the root in base-sixty notation, when Liber abaci is written entirely in base-ten?* We don't know. This sounds like a good research topic!

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(Continued from p. 101)

Never permit him to accept any statement as true which he does not understand. Let him learn not by authority but by demonstration addressed to his own intelligence. Encourage him to ask questions and to interpose objections. Thus he will acquire that most important of all mental habits, that of *thinking for himself*.

Carefully discriminate, in the instruction and exercises, as to which faculty is addressed,—whether that of *analysis* or *reasoning*, or that of *calculation*. Each of these requires peculiar culture, and each has its appropriate period of development. In the *first stage* of arithmetical instruction, *calculation* should be chiefly addressed, and analysis or reasoning employed only after some progress has been made, and then very slowly and progressively. A young child will perform many operations in calculation which are far beyond its powers of analysis to explain thoroughly.

In the exercise of the calculating faculty, the examples should be rapidly performed, without pause for explanation or analysis; and they should have very great variety, and be carefully arranged so as to advance from the simple and rudimental to the complicated and difficult.

In the exercise of the analytical faculty, great care should be taken that the processes do not degenerate into the mere repetition of *formulae*. These forms of expression should be as simple and concise as possible, and should be, as far as practicable, expressed in the pupils own language. Certain necessary points being attended to, the precise from of expression is of no more consequence than any particular letters or diagrams in the demonstration of geometrical theorems. Of course, the teacher should carefully criticise the logic or reasoning, not so as to discourage, but still insisting upon *perfect accuracy* from the first.

The oral or mental arithmetic should go hand in hand with the written. The pupil should be made to perceive that, except for difficulty in retaining long processes in the mind, all arithmetic ought to be oral, and that the slate is only to be called into requisition to aid the mind in retaining intermediate processes and results. The arrangement of this text-book is particularly favorable for this purpose.

(Continued on p. 138)

