Population Growth

Today we will discuss ways to model the growth of a population, be it a population of people, animals, bacteria, or whatever.

**Absolute Growth Rate**

Let \( P(t) \) represent the population at time \( t \). The *absolute growth rate* is the rate of change of the population, or \( P'(t) \). Now we can only use this if a formula is given for \( P(t) \). However, we can use as an *approximation* the *average* absolute growth rate from time \( t = a \) to \( t = b \) by dividing the change in the population by the length of time:

\[
\frac{P(b) - P(a)}{b - a}
\]

(This is just like finding the average velocity by dividing distance traveled by time.)

**Example:** If a population in a small town is 58 in 2008 and 84 in 2010, the absolute growth rate is

\[
\text{persons/year}
\]

This essentially says that on average, the population went up about ___ people per year in this time period.

**Population Formula with Constant Absolute Growth Rate**

Suppose we want to model a population with a function \( P(t) \) on the assumption that the absolute growth rate is constant, \( P'(t) = A \). That is, the population increases or decreases by the same number each time period. The correct formula would be

\[
P(t) = P(0) + At
\]

where \( P \) is the population at time \( t \), \( P(0) \) is the initial population, and \( A \) is the absolute growth rate. Since \( P'(t) = A \) is constant the population growth is linear.

**Example:** If we start a club with 70 members, and agree to recruit 7 new members per week, the number of members \( t \) weeks after we started will be

\[
P(t) = 70 + 7t
\]

It is worth remembering that absolute growth rate is really the rate of change of the population, or the derivative \( P'(t) \). Thus, the fundamental theorem tells us it is easy to find the total change in a population from an absolute growth rate:
Relative Growth Rate

Relative growth represents how a population grew as a percentage of the previous population. Thus, to find the relative rate of growth, we must divide the absolute growth rate by the initial population. Again, let \( P(t) \) represent the population at time \( t \). Then the relative growth rate at time \( t \) is \( \frac{P'(t)}{P(t)} \). Since we do not always have this information, we can use as an approximation the average relative growth rate over the interval from time \( t = a \) to \( t = b \):

\[
\text{Relative growth rate} = \frac{P(b) - P(a)}{b - a}
\]

Relative growth rate is often used to discuss the growth of natural populations, because the rate of growth often depends on how large the population is now.

**Example:** If a population in a small town is 58 in 2008 and 84 in 2010, the relative growth rate is

or about _____% per year.

Population Formula with Constant Relative Growth Rate

If we assume that the average relative growth rate is constant, \( \frac{P'(t)}{P(t)} = r \), then for each increment of \( t \) we must multiply the previous population by \( r \) to find the increase. Then we add the increase to the original population. Suppose for example we have \( t \) in years, and we start with population \( P(0) \). Then in one year, we have population

Then for the next year, we must multiply the new population by the relative growth rate and add it to \( P(1) \):

Then the pattern just repeats. After \( t \) years, we should have population

(This is easy to remember if you think of \( r \) as a percentage; each year we find “one hundred and \( r \) percent” of the previous population, so we multiply by \( 1 + r \) each year.)

**Example:** Suppose a population starts with 1.7 million member in 2000 and increases by 2% per year. The population \( t \) years after 2000 would then be

if we let \( P \) be given in millions.
Finding Total Change from Relative Growth Rate

We noted that since the absolute growth rate is \( P'(t) \), it is easy to find the total change in a population from an absolute growth rate. Since the relative growth rate is \( \frac{P'(t)}{P(t)} \), it is not so easy to see how we might recover total growth from a relative growth rate.

Let’s move totally off subject for a moment, and ask a bizarre question: what is the derivative of \( \ln(P(t)) \)?

\[
\frac{d}{dt} \ln(P(t)) = \]

Notice anything familiar about the answer? In other words, by magic (or a crafty calculus teacher), we have a function which has derivative \( \frac{P'(t)}{P(t)} \). (Such a function is called an antiderivative; we will talk more about these next week.)

Since the fundamental theorem then says that

\[
\int_{a}^{b} \frac{P'(t)}{P(t)} dt =
\]

we therefore observe that

\[
\int_{1907}^{1911} \frac{P'(t)}{P(t)} dt =
\]

using a property of logs for the last step. So then we can get the ratio \( P(b)/P(a) \) by raising \( e \) to the power of the result. Therefore, to find the relative growth of the population, we can first integrate the relative growth rate, then raise \( e \) to the result to get \( P(b)/P(a) \), or the ratio of the ending population to the beginning population.

Example: Suppose we have the following relative growth rates for a population:

<table>
<thead>
<tr>
<th>Year</th>
<th>1907</th>
<th>1909</th>
<th>1911</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Growth Rate</td>
<td>.01</td>
<td>.005</td>
<td>.021</td>
</tr>
</tbody>
</table>

Estimate the relative change in the population from 1907 to 1911. We first estimate the integral:

\[
\int_{1907}^{1911} \frac{P'(t)}{P(t)} dt \approx
\]

Here we used Simpson’s rule with \( n = \) and \( \Delta t = \). \( \frac{P'(t)}{P(t)} \) is just the relative growth rate given in the table. Remember that our result is \( \ln(P(b)/P(a)) \), so we have

\[
\frac{P(1911)}{P(1907)} =
\]

So the population in 1911 was about _____ times as big as it was in 1907. Or to say the
same thing another way, the population increased by about ____ percent between 1907 and 1911. We always __________ to get the % increase or decrease in population.

**Instantaneous vs. Average Relative Growth**

We have discussed essentially two different kinds of constant relative growth in this class. We modeled a constant relative growth rate with the differential equation $y' = ky$, which we said had solution $y = Ce^{kt}$, where $C = P(0)$, the initial population. Today, we came up with a formula for constant relative growth at rate $r$ that looks like $P(t) = P(0)(1 + r)^t$.

So for example, if we want to model a population that grows at 2% per year, if we use the first method we get $P(t) = P(0)e^{0.02t}$, while the second method would give $P(t) = P(0)(1.02)^t$. What’s the difference?

The difference is that the first method assumes an *instantaneous* rate of growth of 2% per year, and the second assumes that the population has grown an *average* of 2% in one year. (It’s the difference between going 45 miles per hour at one instant, and having traveled 45 miles during one hour.) Note that if we plug in $t = 1$ to each formula, we get a different result:

but

Our new method gives exactly 2% growth in every year, but using a differential equation represents a continuous instantaneous growth rate of 2% per year. (That means that we have some increase on the first day, which is compounded on day two, and so on.)

So which should we use? If we are trying to fit a curve to data or want a particular instantaneous growth rate, we might use the differential equation. If we want a particular percentage increase during a given time interval, we might use the equation with constant absolute growth rate instead.

**Summary**

Today, we have

- Calculated absolute $\left( \frac{P(b) - P(a)}{b - a} \right)$ and relative $\left( \frac{P(b) - P(a)}{(b - a)P(a)} \right)$ growth rates.

- Established formulas for a population at time $t$ if the population has a constant absolute ($P(t) = P(0) + P'(t) \cdot t$) or relative ($P(t) = P(0)(1 + r)^t$) growth rate.

- Found the relative change in the population based on its relative growth rate, by integrating the relative growth rate to find $\ln(P(b)/P(a))$, and then solving for $P(b)/P(a)$.

- Discussed the difference between constant instantaneous and constant average relative growth.