Proving Multiply Quantified Statements

Reference:
Vroom, K., & Alzaga Elizondo, T. (2022). Students' Thinking about the Structure of Constructive Existence Proofs. *International Journal of Research in Undergraduate Mathematics Education*, 1-23.

Below are six statements about real numbers m, b, and x.

1. There exist real numbers m, b, and x such that $mx + b = 0$.
2. For all real numbers m, b, and x, $mx + b = 0$.
3. There exists a real number m such that for every real number b there is a real number x such that $mx + b = 0$.
4. There exists a real number m such that for all real numbers x there exists a real number b such that $mx + b = 0$.
5. There are real numbers b and m such that $mx + b = 0$ for every real number x.
6. There exists a real number b and a real number x such that for all real numbers m, $mx+b = 0$.

(a) Can you visualize graphically what each statement is saying? One of them is not true. Which one? Explain why this statement is false.

(b) Match each of the statements that are true with the following proofs.

Proof A:
Let $m = 1$. Then, for any real number b, choose $x = -b$. Therefore,

$$\begin{aligned} mx + b &= 1 \cdot (-b) + b \\ &= -b + b \\ &= 0. \end{aligned}$$

Proof B:
Let $m = 1$. Then, for any real number x, let $b = -x$. So,

$$\begin{aligned} mx + b &= 1 \cdot (-x) + x \\ &= -x + x \\ &= 0. \end{aligned}$$

Proof C:
Fix an arbitrary x. Let $b = 0$. Observe that, when $m = 0$,

$$\begin{aligned} mx + b &= 0 \cdot x + 0 \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Proof D:
Let $b = 0$. Observe that, when $x = 0$,

$$mx + b = m \cdot 0 + 0 = 0$$

for any real number m.

Proof E:
Let $b = 0$, $m = 1$, and $x = 0$. Therefore,

$$mx + b = 1 \cdot 0 + 0 = 0.$$