Geometry of Groups

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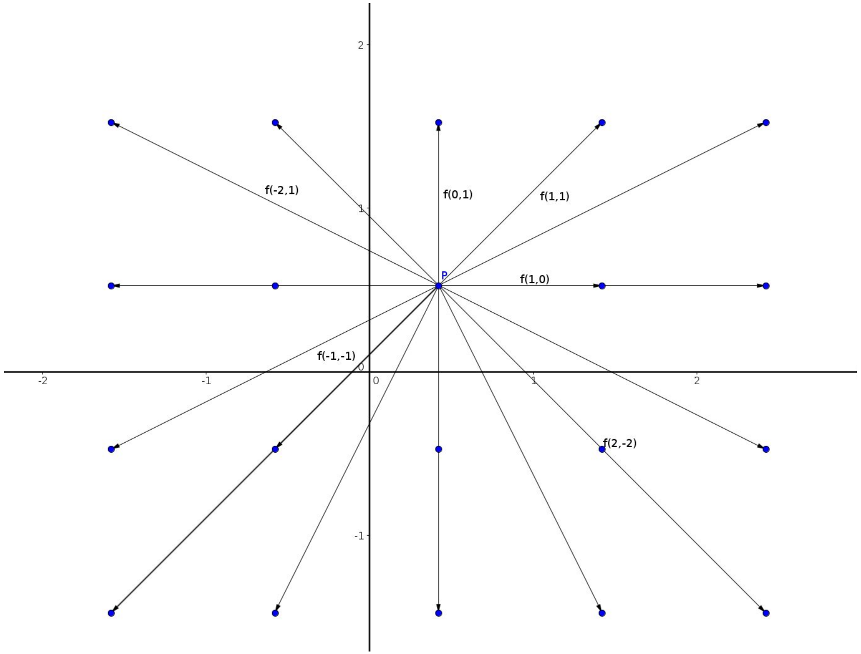
Translations by (m,n)



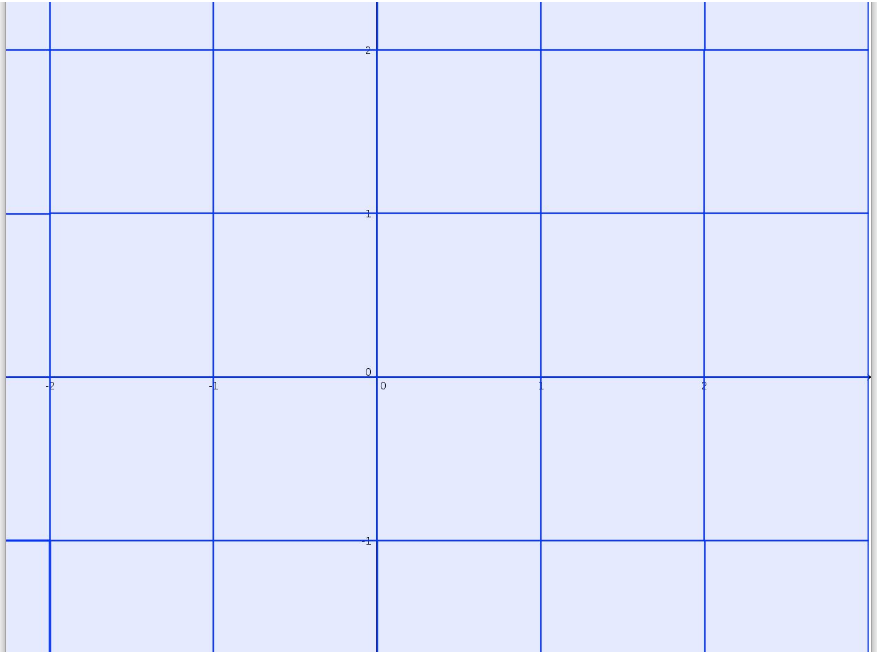
For integers m and n, define a function $f_{(m,n)}$ by

$$f_{(m,n)}(x,y) = (x+m,y+n)$$

The function takes a point (x,y) in the plane, and translates it m units right, n units up.

Write $\mathbf{T^2}$ to mean *all possible* $f_{(m,n)}$ functions. (There are infinitely-many: one for each integer pair)

A fundamental square for $\mathbf{T}^2$

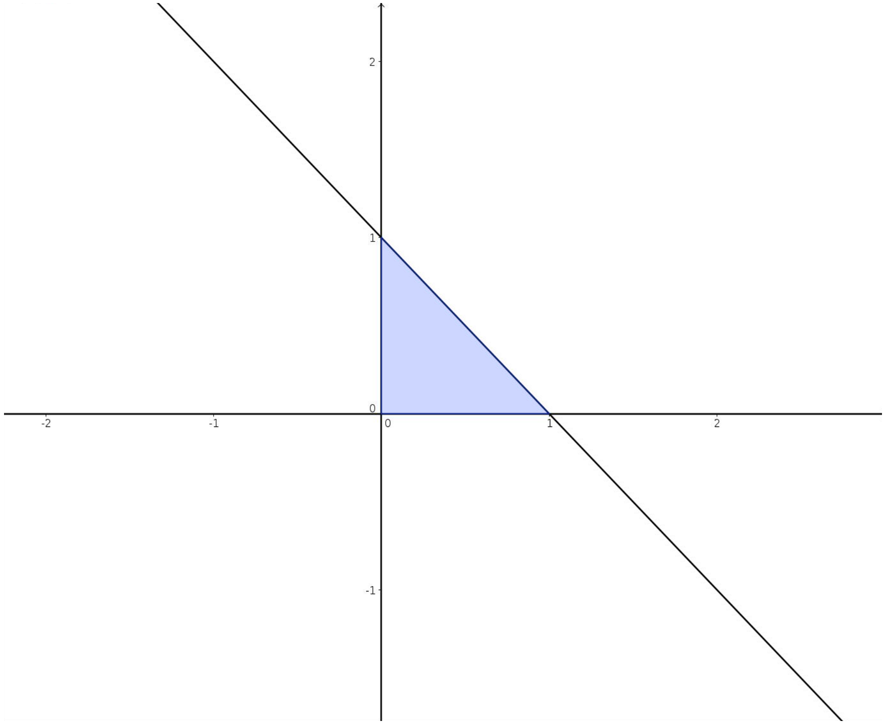


Notice that we can *tile* the plane by applying all $\mathbf{T}^2$ functions to the square with vertices (0,0) (1,0) (1,1) (0,1).

Call this square *fundamental*. We get one copy of the square for each $f_{(m,n)}$.

In this way, we can take something algebraic (functions) and associate it to something geometric (the square).

Reflections in a triangle with angles π/2, π/4, π/4

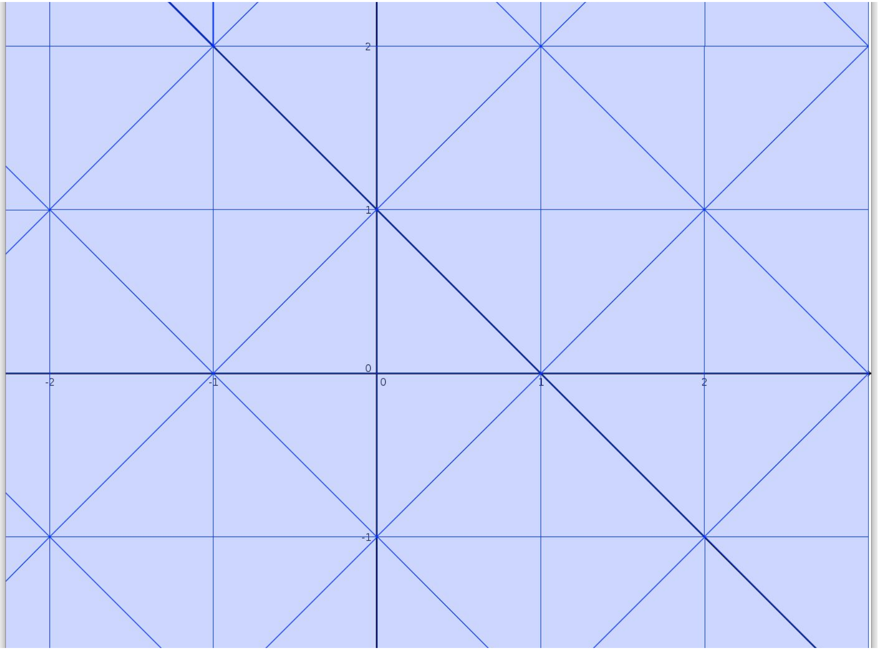


Can we tile the whole plane with triangles if we just keep reflecting over the sides of the triangle?

It turns out that it depends on the angle.

But in this case…

Reflections in a triangle with angles π/2, π/4, π/4

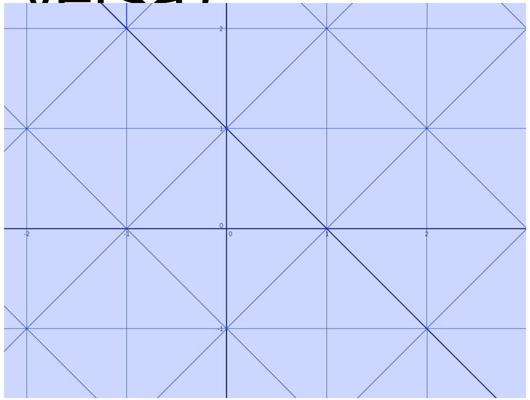


… Yes.

Write $\mathbf{R}^{(2,4,4)}$ to represent *all possible* reflection functions for this triangle. (There are infinitely-many: one for each triangular tile).

In this way, we can build an algebraic object (a collection of functions) from a geometric object.

Can we answer algebra questions with geometry, and vice versa?

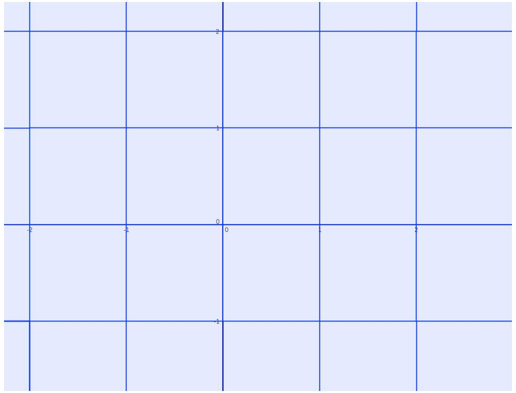


Tiling by $\mathbf{R}^{(2,4,4)}$

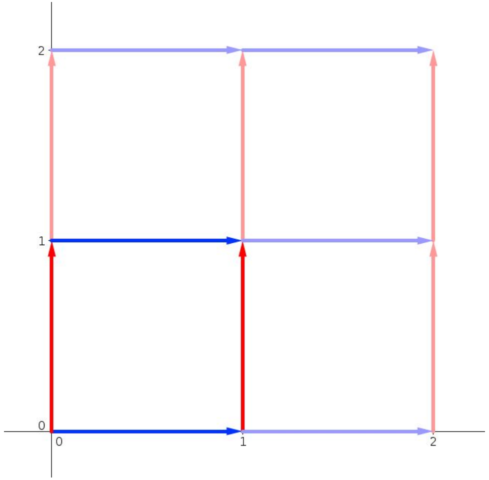
Both pictures have squares in them...

Question:

Are there collections of functions that are common to both $\mathbf{R}^{(2,4,4)}$ and $\mathbf{T}^2$?



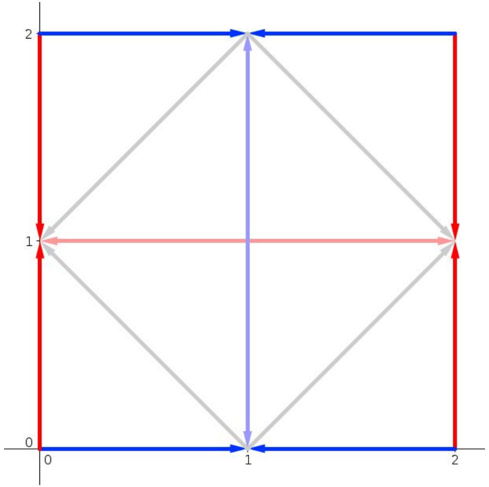
Tiling by $\mathbf{T}^2$



4 fundamental squares for $\mathbf{T}^2$

When marking up the sides of the fundamental square for $\mathbf{T}^2$, every square has left/right and top/bottom sides matching.

When marking up the sides of the fundamental triangle for $\mathbf{R}^{(2,4,4)}$, it takes 8 copies in order to form a square whose left/right and top/bottom edges exactly match.



8 fundamental triangles for $\mathbf{R}^{(2,4,4)}$

This means that both $\mathbf{T}^2$ and $\mathbf{R}^{(2,2,4)}$ share the translations by pairs of *even* integers (m,n). Call this collection of all even translations $\mathbf{ET}^2$.

Collections of matrices – how to "see" the function overlap

T^2

All matrices of the form

1	0	m
0	1	n
0	0	1

where m,n are integers

E^2

All matrices of the form

1	0	m
0	1	n
0	0	1

where m,n are *even* integers

$R^{(2,2,4)}$

All matrices achieved as products of

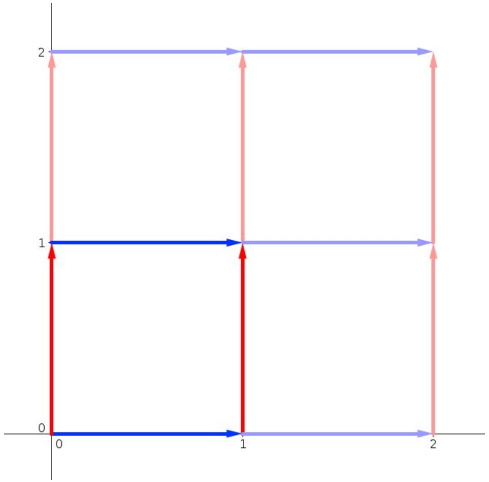
1	0	0
0	-1	0
0	0	1

-1	0	0
0	1	0
0	0	1

and

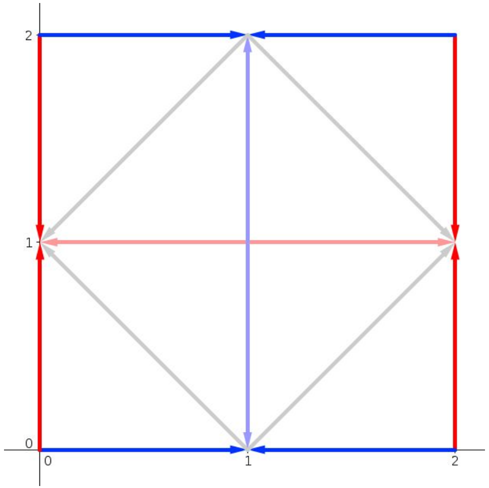
0	-1	1
-1	0	1
0	0	1

More algebra/geometry connections!



These pictures actually tell us something more:

- 4 copies of the $\mathbf{T^2}$ square inside $\mathbf{ET^2}$ square means: ***T²* contains 4 copies of *ET²***
- 8 copies of the $\mathbf{R^{(2,4,4)}}$ triangle inside $\mathbf{ET^2}$ square means: ***R^(2,4,4)* contains 8 copies of *ET²***



My research, generally