Weakly nonlinear behavior of periodic disturbances in two-layer Couette–Poiseuille flow

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The stability of a plane Couette–Poiseuille flow consisting of two layers of different fluids is analyzed using methods of bifurcation theory. The fluids have different viscosities and densities, and there is surface tension at the interface. The center manifold theorem is used to justify the derivation of the final amplitude evolution equation. The nonlinear calculations are carried out with two alternative approaches. One approach is to keep the combined volume flux fixed, and the other is to keep the pressure gradient in the horizontal direction fixed. Numerical results are presented for some Couette flow profiles and a Poiseuille flow profile at low speeds, showing that traveling waves are supported at the interface. A computation at a high speed is also presented. The derivation and numerical results are compared with those of a formal approach, employing multiple scales, which has been used on related problems.

I. INTRODUCTION

A plane Couette–Poiseuille flow consisting of two layers of different fluids is analyzed for stability and bifurcation. The fluids are contained between two infinite parallel plates and have different viscosities and densities, with surface tension at the interface. Gravity and a horizontal pressure gradient are incorporated. In the basic flow, the interface is flat and the flow is in the horizontal direction. The focus is on the calculation of the Stuart–Landau coefficient \( \kappa \). Situations of practical interest cover a wide range of flow parameters, from slow to fast flow;\(^1,2\) examples include the slow extrusion of two molten polymers vertically out of a tube and cooling in air, and the fast pipeline transportation of viscous crude oil with a small amount of water as a lubricant next to the pipe wall. Attention will be focused on slow flow.

Since the earliest experimental results, waves have been observed to travel along the interface.\(^3\) In the linear stability analysis, the spectrum at low speeds consists of an interface mode and an infinite number of “internal” modes, i.e., those that have a counterpart in the one-fluid flow. The internal modes are typically stable at sufficiently low speeds and it is then that the interface mode can cause instability: this was pointed out by Yih,\(^4\) who developed the method for capturing the asymptotic behavior of the interface mode for long wave disturbances. He suggested that the instability of that mode may lead to the establishment of waves on the interface. Indeed, a formal nonlinear analysis to find the amplitude evolution equation with the long wave approximation has been found tractable using a method of multiple scaling.\(^5,6\) It shows that for plane Couette or plane Poiseuille flow of two fluids, surface tension is necessary for a bounded motion. The amplitude evolution equation is then the Kuramoto–Sivashinsky equation, for which numerical and analytical results are given in Ref. 6.

The linear stability of a variety of steady two-layer flows has been examined numerically,\(^8,9,12\) as well as with the short wave\(^1,2,11\) and long wave asymptotics. Other approaches in-

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This paper is dedicated to Professor Daniel D. Joseph for his 60th birthday.

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\(^{14}\) The energy method,\(^15,16\) and large Reynolds number asymptotics.\(^11,17\) These suggest a rule of thumb for linearly stable arrangements in shearing flows at low speeds in the presence of viscosity stratification: place the less viscous fluid in a sufficiently thin layer to stabilize long waves and include enough surface tension to stabilize short waves. The presence of density stratification complicates the rule, as discussed in Sec. IV.

In this paper, the bifurcation from the arrangement with a flat interface is analyzed for the case in which the critical situation consists of a single critical value of the wavelength, with all other wavelengths being linearly stable. The correct manifold approach\(^18\) is used to reduce the nonlinear partial differential equations governing the perturbed motion to an ordinary differential equation. However, there is no version of the center manifold theorem in the literature that can be applied in a straightforward manner to this problem: the variable domain occupied by each fluid and the nonlinearities in the free surface conditions make a full justification tedious. On the other hand, a proof would hinge on coercive estimates for the partial differential equations and these have been derived.\(^19\) Hence the existence of the center manifold is taken for granted in the following, and this ensures that the dynamics close to criticality are determined by the finitely many (in this case, one) critical modes. This idea has been applied to a two-layer Bénard problem,\(^20\) in which the critical eigenvalue has a sixfold degeneracy. There is no such degeneracy in the present problem and the algebra is correspondingly simpler. The procedure is explained in Sec. III. This procedure is standard in bifurcation theory but, as discussed in Sec. III, this is not the only approach that can be made for a nonlinear analysis. An alternative approach is the formal one of Refs. 9 and 21. These approaches are discussed in Sec. III.

The present study is restricted to uniform wave trains, but a comparison with the modulated wave solutions in Ref. 9 yields some unexpected results. In particular, the results indicate that the Landau constant \( \kappa \) depends strongly on
whether it is the pressure gradient that is kept fixed throughout the nonlinear analysis, or whether the combined volume flux is kept fixed.

Numerical results and neutral stability curves at low speeds are presented in Sec. IV. Several Couette flow profiles are used to illustrate the effects of viscosity and density stratifications and surface tension. In particular, a situation with the heavier fluid on top is shown to support traveling waves along the interface. This phenomenon is related to that of the lubrication bearing. The numerical results obtained in Sec. IV are all supercritical bifurcations. For more extreme conditions, there are subcritical bifurcations; e.g., see the Appendix.

II. GOVERNING EQUATIONS

Two fluids of densities \( \rho_i (i = 1, 2) \), and viscosities \( \mu_i \), lie in the \((x^*, z^*)\) plane in layers between infinite parallel plates located at \( z^* = 0, l^* \). Asterisks are used for dimensional variables. The upper plate moves with velocity \((U^*, 0)\) and the bottom plate is at rest. In the basic flow, fluid 1 occupies \(0 < z^* < l^* \) and fluid 2 occupies \( l^* < z^* < l^* \). The velocity of the interface in the basic flow is \((U^*(l^*)), 0)\) and for brevity, we denote \( U^*(l^*) \) by \( U_i \). The velocity, distance, time, and pressure are made dimensionless with respect to \( U_i, l^*/U_i, \) and \( \rho_i U_i^2 \). In Couette–Poiseuille flow, the basic flow has a pressure gradient \(-G^* \) in the \(x\) direction. Reynolds numbers in each fluid are denoted by \( R_i = U_i/l^* \rho_i/\mu_i \) and \( R_2 = U_i/l^* \rho_2/\mu_2 \). There are seven dimensionless parameters: a Reynolds number, say \( R_2 \), the undisturbed depth \( l_i \) of fluid 1, a surface tension parameter \( T = (\text{surface tension coefficient})/(\mu_2 U_i) \), a Froude number \( F \) given by \( F^2 = U_i^2/g l^* \), where \( g \) is the gravitational acceleration constant, a dimensionless pressure gradient \( G = G^*/l^* (\rho_i U_i^2) \), the viscosity ratio \( \varrho = \mu_2/\mu_1 \), and a density ratio \( r = \rho_2/\rho_1 \).

The dimensionless basic flow \((U(x, z), 0)\) is

\[
U(z) = \begin{cases} 
-GR_1 z^2/2 + c_1, & 0 < z^* < l^*, \\
-GR_2 (z - 1)^2/2 + c_2 (z - 1) + U_p, & l^* < z^* < l^*, \\
\end{cases}
\]

where

\[
c_1 = \frac{(1 + GR_1 l_i^2/2)}{l_i}, \quad l_2 = 1 - l_i, \\
c_2 = m(-GR_1 (l_i + c_1) - rGR_2 l_2),
\]

and the upper plate speed \( U_p \) is

\[
U_p = 1 + rGR_2 l_i^2/2 + c_2 l_2.
\]

The basic pressure field \( P \) satisfies \( dP/dx = -G \) and

\[
dP = \begin{cases} 
-1/F^2, & 0 < z^* < l^*, \\
-l/(rF^2), & l^* < z^* < l^*, \\
\end{cases}
\]

(2)

Solutions that are perturbations of the above basic flow are sought. The perturbations to the velocity, pressure, and interface position are denoted by \((u, w, p, h)\), and \(r\), respectively. The Navier–Stokes equations in each fluid yield

\[
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + w \frac{\partial U}{\partial x} + \frac{1}{R_i} \Delta u + \frac{\rho_i \partial p}{\rho_i} \frac{\partial}{\partial x} \\
= -u \frac{\partial u}{\partial x} - w \frac{\partial u}{\partial z}, \quad (3)
\]

\[
\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + \frac{1}{R_i} \Delta w + \frac{\rho_1 \partial p}{\rho_1} \frac{\partial}{\partial x} = -u \frac{\partial w}{\partial x} - w \frac{\partial w}{\partial z}. \quad (4)
\]

Incompressibility reads

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (5)
\]

The boundary conditions are \( u = w = 0 \) at \( z = 1 \). The conditions at the interface are posed at the unknown position \( z = l_i + h(x, t) \). Since the unknown \( h(x, t) \) will be assumed to be small, the method of domain perturbation is used, that is, the interfacial conditions are expanded as Taylor series about \( z = l_i \) and truncated after the cubic term. Continuity of velocity yields

\[
h(U') + \left[ u \right] = -h \left[ \frac{\partial u}{\partial x} \right] - \frac{h^2}{2} \left[ \frac{\partial^2 u}{\partial z^2} \right], \quad (6)
\]

and

\[
\left[ w \right] = -h \left[ \frac{\partial w}{\partial x} \right] - \frac{h^2}{2} \left[ \frac{\partial^2 w}{\partial z^2} \right], \quad (7)
\]

where \([x]\) denotes \( x(\text{fluid 1}) - x(\text{fluid 2}) \). Continuity of shear stress yields

\[
\left[ \frac{2\mu_i}{R_i \mu_i} \frac{\partial w}{\partial z} - p \right] - \frac{T}{mR_i} \frac{\partial^2 h}{\partial x^2} - h \left[ \frac{\partial P}{\partial x} \right] \\
= \frac{2}{R_i} \frac{\partial h}{\partial x} \left[ u \right] + \frac{\partial h}{\partial x} \left[ \frac{2\mu_i}{R_i \mu_i} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \right] - h \left[ \frac{2\mu_i}{R_i \mu_i} \frac{\partial^2 w}{\partial z^2} - \frac{\partial p}{\partial z} \right] - \left( \frac{\partial h}{\partial x} \right) \left[ \frac{2\mu_i}{R_i \mu_i} \frac{\partial u}{\partial x} - p \right] \\
+ \left( \frac{\partial h}{\partial x} \right) \left[ \frac{2\mu_i}{R_i \mu_i} \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial z^2} \right] - \frac{T}{2mR_i} \frac{\partial^2 h}{\partial x^2} \left( \frac{\partial h}{\partial x} \right)^2 + h \left( \frac{\partial h}{\partial z} \right) \left[ \frac{\partial P}{\partial z} \right]. \quad (9)
\]
The kinematic free surface condition yields
\[
\frac{\partial h}{\partial t} + U(t) \frac{\partial h}{\partial x} - w_1 = - \frac{\partial h}{\partial x} (u_1 + hU') + h \frac{\partial w_1}{\partial x} - h \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} - \frac{\partial h}{\partial x} h U'',
\]
where here the subscript 1 refers to fluid 1.

In the linearized stability analysis, we discard terms that are quadratic or higher in the perturbations and seek solutions that are periodic in the \( x \) direction with wavenumber \( \alpha \).
Thus, \( u, w, p, \) and \( h \) are proportional to \( \exp(i\alpha x + \sigma t) \), where \( \sigma \) is a complex-valued eigenvalue and is solved for with the other parameters given. We denote by \( \alpha \), the critical wavenumber where the real part of \( \sigma \) is zero. Asymptotic formulas for \( \alpha > 1 \) and \( \alpha < 1 \), as well as numerical results for linear stability, are discussed in Sec. IV.

### III. THE NONLINEAR PROBLEM

This section is concerned with whether the traveling wave solutions bifurcating from the basic flow at criticality are stable or unstable. Equations (3)-(10) are reduced to an ordinary differential equation by the center manifold approach.\(^{18}\)

Let \( \Phi \) represent the set of unknowns \( (u, w, p, h) \). The nonlinear terms in the equations of motion are retained. Terms up to and including the third order are retained in the interfacial conditions. Thus (3)-(10) have the form
\[
L\Phi = N_2(\Phi, \Phi) + N_1(\Phi, \Phi, \Phi),
\]
where the real linear operator \( L \) has the form \( A + B / dt \). Here \( N_2 \) contains quadratic terms and \( N_1 \) contains cubic terms from the right-hand sides of Eqs. (3)-(10). The notation \( L(\sigma) = A + \sigma B \) will be used. We let \( \lambda \) be the bifurcation parameter: this could be any of the physical parameters, e.g., \( m, \ell, \).
At \( \lambda = 0 \), there is one eigenvalue, the interface modulation, at \( \sigma = -ic, c > 0 \), for \( \alpha = \alpha_0 > 0 \), and a corresponding eigenvalue \( \sigma = ic \) for \( \alpha = -\alpha_0 \), and the rest of the eigenvalues are stable (\( Re \sigma < 0 \)). The eigenfunction with wavenumber \( \alpha \) is denoted by \( z(\lambda) \) and that with wavenumber \( -\alpha \) by \( \xi(\lambda) \), where the overbar denotes the complex conjugate. For \( \lambda > 0 \), \( -i \alpha \) becomes \( -s(\lambda) \) and \( +i \alpha \) becomes the complex conjugate of \( -s(\lambda) \).

The eigenfunctions satisfy \( A\zeta(\lambda) = s(\lambda) B\xi(\lambda) \).

The solution \( \Phi \) can be decomposed, in a manner analogous to the finite-dimensional case, into the decomposition of \( C^\alpha \) by the \( n \) linearly independent eigenfunctions of an \( n \times n \) matrix with distinct eigenvalues:
\[
\Phi = Z_\alpha \Phi + Z_{\bar{\alpha}} \Psi + \Psi.
\]
Here, the \( Z(t) \) is the complex-valued amplitude function and \( \Psi \) denotes a linear combination of the other eigenfunctions, and possibly generalized eigenfunctions. The adjoint eigenfunction with wavenumber \( \alpha \) is denoted by \( b(\lambda) \) and satisfies
\[
(b, L(-s(\lambda))) \Phi = 0,
\]
for every \( \Phi \), where the parentheses denote an inner product. For computations, the Euclidean inner product is used, defined as follows. In the linear calculations, the equations are discretized in the \( z \) direction using a spectral method, namely, the Chebyshev–Tau method.\(^{22}\) This method approximates discrete eigenvalues belonging to \( C^\alpha \) eigenfunctions with infinite-order accuracy. Thus the functions that arise in the nonlinear calculations are readily available as series in the Chebyshev polynomials \( T_n(z) \). Hence
\[
(g_1, g_2) = \frac{\int_A \exp(i(-\alpha_1 + \alpha_2) x) dx}{\int_A dx} \times \sum_{i=0}^N g_{1i} g_{2i},
\]
\[
g_1 = \sum_{i=0}^N g_{1i} T_i(z_i) \exp i\alpha x,
\]
\[
g_2 = \sum_{i=0}^N g_{2i} T_i(z_i) \exp i\alpha x,
\]
\[
z_1 = 2\alpha / \ell - 1, \quad z_2 = 2(\alpha - 1)/\ell + 1,
\]
where \( A \) denotes one wavelength [0, 2\pi/\alpha \]. Note that the complex conjugate is placed on the first member of the inner product. In the actual computations, \( \lambda = 0 \) and the interaction equations involve the eigenfunction and adjoint eigenfunction of the linearized problem at criticality. Since the \( -\alpha_1 + \alpha_2 \) then appear as an integral multiple of the critical wavenumber \( \alpha_0 \), the inner product vanishes unless \( -\alpha_1 + \alpha_2 = 0 \).

Since \( L(-s) = A - sB \), the equation satisfied by the adjoint eigenfunction is
\[
A^* b(\lambda) = s(\lambda) B^* b(\lambda).
\]
The adjoint eigenfunction is normalized by
\[
(b, B\zeta) = 1,
\]
so that we also have \( (\bar{b}, B\bar{\zeta}) = 1 \). Since the eigenfunction satisfies \( (A - s(\lambda) B) \zeta = 0 \), and \( A \) and \( B \) are real operators, \( (A - s(\lambda) B) \bar{\zeta} = 0 \). Hence \( 0 = (b, (A - s(\lambda) B) \bar{\zeta}) \) and
\[
A^* (b\zeta) - s(\lambda) (b, B\zeta),
\]
using (15) for the adjoint, this yields \( 0 = s(\lambda) (b, B\zeta) - s(\lambda) (b, B\zeta) \).

Now, \( (b, B\zeta) = 0 \).

Similar manipulations yield \( (\bar{b}, B\bar{\zeta}) = 0 \). Further, \( \Psi \) is a linear combination of eigenvectors belonging to eigenvalues \( \sigma \) that are not equal to \( -s(\lambda) \). If \( \psi \) is such an eigenvector and \( L(\sigma) \psi = 0 \), then \( 0 = (b, L(\sigma) \psi) \), where \( L(\sigma) = A + B \sigma \), so that \( 0 = (b, A \psi) + (b, B \psi) \) and using (15), \( 0 = (s(\lambda) B^* b, \psi) + (s(\lambda) B, \psi) = s(\lambda) (\sigma) (b, B\psi) \). Hence \( (b, B\psi) = 0 \). Since \( \Psi \) is a linear combination of such \( \psi \)’s,
\[
(b, B\Psi) = 0.
\]

Also, \( 0 = (A^* - s(\lambda) B^* b, \Psi) = (b, A\Psi) - s(\lambda) (b, B\Psi) \) and together with (18), we have
\[
(b, A\Psi) = 0.
\]

The inner product of the adjoint with \( B\Phi \), where the expression for \( \Phi \) in (12) is used, together with (18), yields \( (b, B\Phi) = Z(b, B\bar{\zeta}) + Z(b, B\zeta) \). Using (17) and the orthonormality condition (16),
\[
Z = (b, B\Phi).
\]
The inner product of the adjoint with \( L\Phi \) is
\[
(b, (A + B d/dt) \Phi) = (b, A\Phi) + (b, B d\Phi/dt).
\]

last term is \(dZ/dt\). The second to last term is \((A \ast_b \Phi)\), which, when using the equation satisfied by the adjoint, becomes \((s(\lambda)b, B\Phi) = s(\lambda)Z\). This, together with (11) for \(L\Phi\), yields

\[
\frac{dZ}{dt} + s(\lambda)Z = (b, N_2) + (b, N_3). \tag{21}
\]

The right-hand side of this equation will be shown to be of the form \(\kappa Z^2 + Z\). The purpose of this analysis is to calculate \(\kappa\). If the real part of \(\kappa\) is negative, then the bifurcating solution is supercritical and the travelling wave solution would be stable for small amplitudes. If the real part of \(\kappa\) is positive, then the bifurcating solution would be unstable.\(^{18}\) Discussion about the equilibrium amplitude is found in Ref. 23.

The center manifold theorem essentially states that the dynamics close to the critical situation \(\Phi = 0\) are dominated by the finitely many critical modes. The center manifold is of the form \(\Psi = \tau(z\bar{Z})\). All small periodic solutions lie on the center manifold and the stability of a small periodic solution is determined by its stability within the center manifold. All solutions with initial data on the center manifold remain on the center manifold as long as they remain small.\(^{18}\) The asymptotic expansion of the center manifold for small solutions yields \(\Psi = \Psi_1 + \ldots\) higher-order terms, where

\[
\Psi_1 = Z^2\eta + Z\bar{Z}\chi + \bar{Z}Z\chi + \bar{Z}Z\eta. \tag{22}
\]

Here, \(\chi\) represents the distortion to the mean flow and \(\eta\) is the second harmonic.

In order to evaluate the functions \(\eta\) and \(\chi\), an equation that yields the leading terms in \(\Psi\) in terms of the leading terms in \(\Phi\) is required. To this end, note that

\[
\Phi = (b, B\Phi)\xi + (b, B\Phi)\xi + \Psi. \tag{23}
\]

Analogously, any real-valued \(f\) representing the right-hand side of Eq. (11) can be written as

\[
f = (b, f)B\xi + (b, f)B\xi + g, \tag{24}
\]

where \((b, g)\) is defined by Eq. (16). A projection operator \(\Pi\) is defined so that \(\Pi\Phi\) equals the first two terms of the right-hand side of (23):

\[
g = (I - \Pi)f. \tag{25}
\]

Since

\[
L\Phi = (s + \frac{d}{dt})ZB\xi + (\bar{s} + \frac{d}{dt})Z\bar{B}\xi + L\Psi, \tag{26}
\]

the application of \((I - \Pi)\) to (26) yields

\[
L\Psi = (I - \Pi)(N_2 + N_3), \tag{27}
\]

\[
(A + B\frac{d}{dt})\Psi = N_2 + N_3 - (b, N_2 + N_3)B\xi \tag{28}
\]

The leading terms from (22) are substituted into the left-hand side of the above equation and the leading terms in \(\Phi\) are substituted into the nonlinear terms on the right-hand side. The coefficient of \(Z^2\) in the resulting expression yields

\[
(A - 2s(\lambda)B)\eta = \text{coefficient of } Z^2 \text{ in } N_2 - (b, N_2)B\xi \tag{29}
\]

and the coefficient of \(Z\bar{Z}\) yields

\[
A\chi - s(\lambda) + \bar{s}(\lambda)\chi = \text{coefficient of } Z\bar{Z} \text{ in } N_2 - (b, N_2)B\xi \tag{30}
\]

Here, the definition of \(N_2\) is extended to cover the case of unequal arguments as follows. Here \(N_2(f, g)\) denotes \(0.25(N_2(f + g, f + g) - N_2(f - g, f - g))\). For example, if \((b, \Phi, \Phi) = \Phi(d\Phi/dt)\), then \(N_2(f, g) = 0.5(f df/dz + g dg/dz)\). This represents the average of the two "permutations" of the quadratic expression. Similarly, \(N_2(f, g, h)\) is defined as \(\frac{1}{6}\) of the six possible permutations of the cubic expression. Many of the inner products vanish. For example, since \(\xi\) and \(b\) are proportional to \(\frac{i\omega}{\kappa}\) \(x\), \(N_2(\xi, \xi, \xi)\) is proportional to \(\frac{2i\omega}{\kappa}\) \(x\) so that \((b, N_2(\xi, \xi))\) and \((b, N_2(\xi, \xi))\) both vanish. Therefore

\[
(A - 2s(\lambda)B)\eta = N_2(\xi, \xi, \xi), \tag{31}
\]

where, in actual computations, \(\eta\) is proportional to \(\frac{2i\omega}{\kappa}\) \(x\) and \(\lambda\) is set to zero. Similarly, the equation for \(\chi\) simplifies to

\[
A\chi - s(\lambda) + \bar{s}(\lambda)\chi = N_2(\xi, \xi, \xi), \tag{32}
\]

and \(\chi\) has no \(x\) dependence. At criticality, \(s(0) + \bar{s}(0)\) is equal to \(N_2(\xi, \xi, \xi)\). Since \(\chi\) has no \(x\) dependence, its component \(w\) satisfies \(dw/dz = 0\) by incompressibility. Since \(w = 0\) at \(z = 0, 1, w = 0\) in the entire domain. Denote the quadratic terms on the right-hand sides of Eqs. (3) and (4) by \(f_1\) and \(f_2\) in fluid 1, and by \(f_3\) and \(f_4\) in fluid 2, respectively. Denote the quadratic terms on the right-hand sides of Eqs. (6)–(10) by \(f_5\), \(f_6\), \(f_7\), \(f_8\), and \(f_9\), respectively. We note that \(f_1\) in \(N_2(\xi, \xi, \xi)\) vanishes. Putting \(dU/dx = 0\), the velocity component \(u\) in \(\xi\) satisfies

\[
\begin{aligned}
\frac{1}{R_1} \frac{d^2u}{dz^2} &\text{ in fluid 1 } f_1, \\
\frac{1}{R_2} \frac{d^2u}{dz^2} &\text{ in fluid 2 } f_2,
\end{aligned} \tag{33}
\]

and

\[
\frac{\mu}{R_1} \frac{d^2u}{dz^2} + [u] = f_3. \tag{34}
\]

There is no condition on \(h\) and it could be set to anything. However, there is a null space for the linear operator \(L\) with eigenvalue 0 that affects the component in \(\Phi\) that does not depend on \(x\). One can solve the above problem for \(u\) with zero right-hand sides, and find an eigenfunction with \(h = \text{constant}\). This is a mode that simply shifts the interface up or down. Because this violates the condition that the volumes of the fluids are given, we must put

\[
h = 0 \tag{35}
\]

for \(\chi\). Therefore the problem for \(p\) becomes

\[
\frac{dp}{dz} \text{ in fluid 1 } f_2, \tag{36}
\]

\[
r \frac{dp}{dz} \text{ in fluid 2 } f_4, \tag{37}
\]

\[
- [p] = f_6. \tag{38}
\]
The problem for $p$ requires an additional condition. This corresponds to the fact that the operator $L$ has a null space consisting of $p = \text{const}, u = w = h = 0$, and the addition of an arbitrary constant to the pressure does not affect the problem. To rule out this eigenfunction, $p$ is made unique by setting

$$p \text{ for fluid } 1 = 0, \text{ at } z = l_1,$$

so that Eq. (37) becomes

$$p \text{ for fluid } 2 = f_0, \text{ at } z = l_1.$$  \hfill (39)

Finally, the functions involved in (22) have been calculated so that the solution $\Phi$, known up to quadratic terms in the neighborhood of criticality, can be substituted into the nonlinear terms on the right-hand side of (21). Again, many of the inner products vanish, and the final equation for the amplitude function is

$$\frac{dZ}{dt} + s(\lambda)Z = \kappa |Z|^2 Z,$$

$$\kappa = (b_4 N_2 (\zeta; \chi) + 2 N_2 (\zeta, \eta) + 3 N_3 (\zeta, \xi, \zeta)).$$  \hfill (40)

This is the Landau equation and $\kappa$ is the Stuart–Landau coefficient [see Sec. 8.3 (iii) of Ref. 23].

The rest of this section concerns comparisons made with known results for single-fluid flow$^{24-26}$ and with the results of Blennerhasset$^7$ for a two-layer system. Agreement for the single-fluid case is found to be good. An appendix is devoted to the comparison with the two-layer system.$^9$

The computer code to carry out the calculation of $\kappa$ was checked against the known values for the one-fluid Poiseuille flow. Two entries in Table 1 in Ref. 24 were considered. The first is their $R = 4000, \alpha = 1.0$. Converting to the present notation, $R_1 = 8000, \alpha = 2.0$. Our other parameters were chosen to be $l_1 = 0.5, m = r = 1$, and $T = 1/F^2 = 0$. With 25 Chebyshev modes in each fluid, the resulting least stable shear mode eigenvalue is $\sigma = (-0.9892 E - 2, -0.55708)$. The complex parameter $2\beta$, listed in their Table 1, is related to our $\kappa$ by $\kappa = -16\beta [w(l_1 = 0.5)]^2/\alpha^2$, where the present code yields $|w(l_1 = 0.5)| = 0.7402$. This would imply $\kappa$ should be $(21.149, -110.346)$. However, Pekeris and Shkoller carry out the nonlinear part of the calculations by putting the growth rate $Re \sigma$ equal to 0 even when it is not. This is an approximation they use for regions close to criticality. When this adjustment is made in the present computations, $\kappa = (21.135, -110.180)$. Without this adjustment, the computed value is $(18.57, -113.30)$. A second check with their Table 1 is at their $R = 6000, \alpha = 1.0$. Our parameters are $R_1 = 12000, \alpha = 2.0, l_1 = 0.5, \sigma = (0.646 E - 3.051963)$, and $\kappa = (23.59, -147.69)$ with the adjustment that $Re \sigma = 0$, and $(23.66, -147.31)$ without. Their $\beta_1/\beta_2$ is 6.24 while our $\kappa_1/\kappa_2$ is 6.26.

The next discussion concerns the results of Blennerhasset$^7$ for a two-layer system. While the present analysis is restricted to uniform wave trains, comparison with the modulated wave solutions of Ref. 9 provides some unexpected results. In particular, the results below (see also the Appendix) indicate that the Landau constant $\kappa$ is strongly dependent on the normalization chosen for the distortion to the mean flow in the case of uniform waves, and further that it is different from the value of $\kappa$ obtained when modulation is allowed.

The Appendix shows the results for some of the data corresponding to Tables 2–5 and 7–10 of Ref. 9. A comparison reveals disagreement in the nonlinear calculations. There are two reasons for the difference, apart from the possibility of bugs in the computer code. The first reason is the same one that explains the difference between the one-fluid results of Ref. 24 and Refs. 25 and 26, and lies in the calculation of $\chi$, the distortion to the mean flow. These works represent two different approaches. In our case, Eqs. (30)–(39) show that the pressure gradient in the $x$ direction in the entire nonlinear analysis is fixed, while the combined volume flux is not fixed. In Refs. 9, 25, and 26, the streamfunction is used, and is set to zero at the walls. This fixes the combined volume flux in the entire analysis, while the pressure gradient in the $x$ direction is not fixed. In our notation, the latter approach is implemented by changing the calculation of the function $\chi$ in Eqs. (30)–(33), for which the unknowns would then be the streamfunction and a constant $dp/dx$. Let $\psi$ denote the streamfunction for $\chi$. Then (30) and (31) are replaced by

$$- \frac{1}{R_1} \frac{d^3 \psi}{dx^3} + \frac{\partial p}{\partial x} \text{ in fluid } 1 = f_1,$$

$$- \frac{1}{R_2} \frac{d^3 \psi}{dx^3} + \frac{\partial p}{\partial x} \text{ in fluid } 2 = f_2.$$  \hfill (41)

Equations (32) and (33) are replaced by

$$\frac{\partial \psi}{\partial z} = 0, \text{ at } z = 0, 1,$$

$$\left[ \frac{\partial \psi}{\partial z} \right] = f_0,$$

$$\mu_{l_1} \frac{\partial^2 \psi}{\partial z^2} = f_0.$$  \hfill (43)

In addition,

$$\psi = 0, \text{ at } z = 0, 1,$$

and the Taylor expansion around $z = l_1$ for the continuity of the streamfunction across $z = l_1 + h(x,t)$ reads

$$\psi = f_{10},$$

where $f_{10}$ is the quadratic term from $N_2 (\zeta, \zeta, \zeta)$ denoted by

$$f_{10} = - h [u] - h^2 \frac{dU}{dz}.$$  \hfill (45)

Equations (34)–(39) stay as is. This approach was also programmed in order to check our code with the one-fluid results of Refs. 25 and 26 at the first criticality of plane Poiseuille flow: $R_1 = 11544.44, \alpha = 2.0412, l_1 = 0.5, m = r = 1$, and $T = 1/F^2 = 0$. Our $\kappa$ is related to the $k = (30.8, -173)$ in Eq. (2.35) of Ref. 26 by $\kappa = 2\kappa [w(l_1 = 0.5)]^2/\alpha^2$, where we find $|w(l_1 = 0.5)| = 0.7457$. This conversion yields $\kappa = (32.9, -184.93)$, compared with the present calculation, with 35 Chebyshev modes in each fluid, of $(33.0, -184.6)$. On the other hand, when the pressure gradient is fixed, $\kappa = (31.7, -153.53)$, so that either approach yields a similar result for the real part of $\kappa$, which is of inter-
est. These nonlinear calculations were also checked with \(l_i = 0.3\).

The one-fluid combined Couette–Poiseuille flow of Table 4 in Ref. 25 was checked for order. The value of \(b_0^{(2)}/a_0^{(2)}/u_c = 0.3\) in their Table 4 is 2.006, and this corresponds to \(\kappa_2/\kappa_c\). The same ratio from Table A5 of Ref. 9 is 2.06. Our computation with fixed volume flux, using 35 Chebyshev modes in each fluid at \(R_l = 38929.371\), \(G = 0.0001712\), \(l_i = 0.5\), \(\alpha_c = 0.958\), \(m = r = 1\), and \(R = l_1/F^2 = 0\) yields \(\sigma = (0.433E - 5.0, -0.222)\), \(\kappa = (193.56, -396.77)\), and \(\kappa_c/\kappa\approx 2.05\). Thus the pressure gradient is fixed, \(\kappa = (192.91, -384.18)\). Thus there is no much difference in \(Re\,\kappa\) between the two approaches. This is not always true for two-fluid problems as evidenced in the Appendix; e.g., Tables VIII and X contain situations where the \(Re\,\kappa\) from the two approaches have opposing signs.

The second reason for the difference between Blewett-Hassett’s approach and the present one is the fact that he has a dispersion relation Eq. (2.37), whereas there is none in the present work. A standard idea in bifurcation theory used here is that the wavenumber is considered to be fixed and the harmonics are taken into the nonlinear solution. This gives rise to the function \(\chi\) which has no dependence on \(x\). This leads to Eq. (34). This approach is different from that of Ref. 9, which uses a formal method, employing multiple scales for describing amplitude modulations. Our \(\chi\) corresponds to his \(\phi_{02}\) in Eq. (2.66). His \(\phi_{02}\) has an \(x\) dependence, where the wavenumber is assumed to be small but nonzero, and then the limit as the wavenumber approaches zero is taken. For every nonzero wavenumber, his interface has a nonzero amplitude \(h\), in contrast with our Eq. (34). This implies that for small wavenumbers, the volume ratio of the two fluids is not equal to the given original value over large lengths of the channel, but on average, the volume ratio is preserved. His kinematic condition for \(\phi_{02}\) is, from Eq. (10), essentially

\[wh + iaU(l_i)\phi = -ia\phi + \text{nonlinear terms},\]

where \(\phi\) is the streamfunction. At small \(\alpha\), \(\sigma\) is therefore \(O(\alpha)\) and \(h = O(\phi)\); so that the limit \(\alpha \to 0\) yields a nonzero \(h\). This may be one of the reasons for the numerical difference between his results and the present ones. (However, in the single-fluid case, this issue does not arise and the comparisons made above with Ref. 26, where amplitude modulation in space is also allowed, gives satisfactory agreement.)

The computations presented in the tables and figures of this paper have been tested for convergence. A consistency test has been performed for the case where the layer densities are the same (or gravity is absent) by interchanging the layers as follows. In the original data, let \(\rho_1, \rho_2, l_i, \) and \(l^*\) be fixed. Values for \(\mu_1\) and \(\mu_2\) are chosen, and this yields the dimensionless upper plate speed \(U_2\). A choice for \(R_l\) yields the value of the interfacial speed \(U_i\) since \(U_i = R_l\mu_i/\rho_i l^*\). A choice for \(T\) yields the value of the dimensional surface tension coefficient \(\gamma_{02}U_i\). The dimensionless speed of the upper plate is \(U_2^* = U_2U_i\). In order to interchange the layers, the values for \(\mu_1\) and \(\mu_2\) are interchanged. The plate speed \(U_2^*\) remains as before and this yields a new value for \(U_i\), and consequently for \(R_l\) and \(T\).

Since \(\sigma(t) = (\sigma U_i)/l^*/l^*\), the quantity \(\sigma U_i\) is compared for the two sets of data. It is found that in the linear calculations, \(Re\,\sigma\,U_i\) stays the same. However, \(Im\,\sigma\,U_i\) changes for the following reason. If the flow field is simply turned upside down (e.g., if \(l_i = 0.5\), the reflection across the centerline), then the lower plate would move and the results should remain the same. In the above transformation of “interchanging layers,” not only have the fluids been exchanged but the shear has been reversed and a uniform flow has been superposed (in order to keep the lower plate fixed). This accounts for the difference in the \(Im\,\sigma\,U_i\), and in the nonlinear calculations, in the reversal of the sign of \(Im\,\kappa\). The values for \(\kappa\) for the two sets of data then agree to within a constant multiple, which arises from the normalizations involved in the functions present in Eq. (40).

In conclusion, it is noted that there is no “unique” way of carrying out the nonlinear analysis, and it is difficult to say which approach is realistic to practical situations.

IV. NEUTRAL STABILITY CURVES AND NUMERICAL RESULTS

The asymptotic analysis of the interfacial eigenvalue for short wave disturbances was first carried out by Hooper and Boyd.12 They showed that short wave instability is possible if, for example, there is not enough surface tension, or the density stratification is adverse. In particular, the presence of viscosity stratification alone causes short wave instability. Their analysis proceeds by rescaling the vertical variable \(z\) to \(\eta\), such that \(z - l_i = \eta/\alpha_0\) and setting \(\eta = O(1), \alpha_0 \gg 1\). The velocity is expanded in a series in \(1/\alpha_0\)

\[\sigma = iaU(l_i) - c_0 + c_1/\alpha_0 + c_2/\alpha_0^2 + \cdots\]

(46)

for \(\alpha_0 \gg 1\). The boundary conditions at \(z = 0,1\) become conditions on the decay of the velocity as \(\eta \to \infty\). In the normal stress condition, one assumes a distinguished limit: \(\sigma^T = O(1), \sigma[\partial \phi / \partial x] = O(1), \) where \(P\) is the basic pressure field given by Eq. (2). The application of this formal method to two-layer flows is described in detail in Refs. 10, 12, 27, and 28. The results are \(c_0 = c_1 = 0\) and

\[\sigma = iaU(l_i) - mR_1 \left( \frac{1}{m^2 + \frac{GR_1}{2}(1 - m)} \left( \frac{1}{m^2 + \frac{GR_1}{2}(1 - m)} \left( \frac{1}{1 - m} \right) \right) \right).\]

(47)

Stability to long wave disturbances is necessary but not sufficient for linear stability. The asymptotic analysis of the interfacial mode for long wave disturbances was first carried out by Yih4 for two-layer Couette flow and for the two-layer Poiseuille flow. The results for combined Couette–Poiseuille flow are implicitly contained in Sec. 2.1.1 of Ref. 9; these results will be made explicit below. Details of the procedure involved in this type of analysis are available elsewhere10; the following is a summary.

Let \(\psi(z) \exp(i \alpha x + \alpha t)\) be a streamfunction. The streamfunction and eigenvalue are expanded for small \(\alpha\) and \(\alpha R_l: \sigma = \sigma_0 + \sigma_1 + \cdots\). At the leading order, \(\psi(z) = C_1 z^2 + D_1 z^3 \) in fluid 1 and \(\psi(z) = C_2 (z - 1)^2 + D_2 (z - 1)^3 \) in fluid 2, where
\[ D_1 = \frac{n^2 - m}{[3ml_1 + l_2(2m - n^2)]}, \]
\[ n = l_1/l_2, \quad l_1 = 1 - l_2, \]
\[ D_2 = ml_1, \]
\[ C_2 = n^2 + D_1l_2(n^2 + m), \]
and we may choose
\[ C_1 = 1. \]
At this order the eigenvalue is imaginary:
\[ \sigma_0 + i\alpha U(l_1) \]
\[ \sim \frac{1}{(l_1^2 + D_1l_2^2)} \left[ U^* U \right] \left[ 2l_1 + 3D_1(l_1^2 - ml_2^2) \right. \]
\[ \left. + 2C_2l_2 \right] \]  
(53)
At the next order, \( \psi'' = i\alpha R_l [(U + \sigma_0)\psi'' - \sigma \psi] \).
Thus, in fluid 1,
\[ \psi(z) = E_1z^2 + F_1z^3 + i\alpha R_l h_1(z), \]
where
\[ h_1(z) = a_2z^2 + a_3z^3 + a_5z^4 + a_7z^5, \]
\[ a_0 = (GR_1l_1^2/2 - c_1l_1 + \alpha_0)/12, \]
\[ a_2 = [c_1 + 3D_1(3Gr_1l_1^2/2 - c_1l_1 + a_0 /i\alpha)] /60, \]
\[ a_5 = c_1D_1/60, \quad a_7 = -2GR_1l_1^3/420, \]
\[ \sigma_0 = \sigma_0 + i\alpha U(l_1). \]
(58)
Here, \( \sigma_0 \) is found from Eq. (53) and the small script \( c_1 \) is defined in Eq. (1). Similarly, in fluid 2,
\[ \psi(z) = E_2(z - 1)^2 + F_2(z - 1)^3 + i\alpha R_l h_2(z), \]
(59)
where
\[ h_2(z) = b_4(z - 1)^4 + b_5(z - 1)^5 \]
\[ + b_6(z - 1)^6 + b_7(z - 1)^7, \]
and \( b_2 \) are defined analogously to \( a_2 \). We may choose
\[ E_2 = i\alpha R_2. \]
(60)
Continuity of shear stress yields
\[ 2E_1 + 6F_1l_1 + i\alpha R_l h_1^z(l_1) \]
\[ = (1/m) \left[ 2E_2 - 6F_2l_2 + i\alpha R_l h_2^z(l_1) \right]. \]
(61)
The normal stress condition yields
\[ 6F_1l_1/r = 6F_2l_2/r + iaF, \]
(62)
where
\[ f = \left[ \frac{1}{r} \right] \left( U + \frac{\sigma_0}{i\alpha} \right) (2l_1 + 3D_1l_1^2) \]
\[ + (l_1^2 + D_1l_1^4) \left( \frac{1}{r} - 1 \right) \]
\[ \times \left[ (U + \frac{1}{F^2(U + \sigma_0 / i\alpha)}) - h_1(z)(l_1) + \frac{h_2^z(l_1)}{r} \right]. \]
(64)
Here, \( \sigma_0 \) is given by Eq. (53) and is imaginary. Continuity of horizontal velocity yields
\[ (U + \sigma_0 / i\alpha) [2E_1l_1 + 3F_1l_1^2 + i\alpha R_l h_1^z(l_1) + 2E_2l_2 \]
\[ - 3F_2l_2^2 - i\alpha R_l h_2^z(l_1)] - [E_1l_1^3 + F_1l_1^4 \]
\[ + i\alpha R_l h_1(l_1)] [U^* U] \]
\[ = (\sigma_0 / i\alpha) [2E_1l_1 + 3D_1(l_1^2 - ml_2^2) + 2C_2l_2], \]
(65)
where \( \sigma_0 \) is the correction to the eigenvalue and is real. Continuity of vertical velocity yields
\[ E_1l_1^3 + F_1l_1^4 + i\alpha R_l h_1(l_1) \]
\[ = E_2l_2^3 + F_2l_2^4 + i\alpha R_l h_2(l_1). \]
(66)
We substitute for \( F_1 \) using (63) in (62) and (66). We eliminate \( E_1 \) from (62) and (66) and use (61) to find \( F_2 \). Equation (62) then yields \( E_2 \). Finally, (65) yields the real-valued term \( \sigma_0 \).

Next we discuss the calculation of the interfacial eigenvalue for the general case without the asymptotic approximations. The neutral stability curves presented in the figures of this paper are determined numerically. The full linearized equations are discretized with the Chebyshev–Tau method, which has infinite-order accuracy for \( C \)–eigenfunctions. The accuracy of the computer code in calculating the interfacial eigenvalue was checked against the long wave formulas of Yih, the short wave formula of Hooper and Boyd, and Tables 2–5 of Blennerhasset (see the Appendix). The code was also checked in the case when the fluids are the same, to retrieve the one-fluid modes, at the criticality of plane Poiseuille flow and at the higher Reynolds number listed in Table 5 of Ref. 22.

For plane Poiseuille flow, neutral stability curves for the interfacial and shear modes have been presented by Yiantsios and Higgins. Their Fig. 2(b) gives the neutral stability diagram for long wave disturbances plotted in the \( n \)-plane where \( n \) is their thickness ratio. This is of interest because a part of such a diagram may coincide with the neutral stability diagram for all wavelengths when surface tension is large. Their other neutral stability curves for the interfacial mode are plotted in the \( \alpha-n \) or the \( \alpha \)-Reynolds number plane. The shear mode instability takes place at relatively higher Reynolds numbers and will not be pursued in this paper.

In Sec. VI of Ref. 10, Squire's theorem is examined. The following points are noted. There are two parts to the theorem. The first portion of Squire's theorem is that there is a Squire's transformation for the two-layer problem. Thus if there is a three-dimensional instability proportional to \( \exp(i\alpha x) \) at a Reynolds number, say \( R \), then there is a corresponding two-dimensional instability proportional to \( \exp(i\alpha x) \) at a lower Reynolds number \( \tilde{R} \). and Squire's transformation:
\[ \tilde{\alpha} = \alpha^2 + \beta^2, \tilde{R} = R/\tilde{\alpha}. \] This \( \tilde{\alpha} \geq \alpha \) and \( \tilde{R} < R \). An alternative way of stating this is that the critical Reynolds number for the three-dimensional problem is higher than for the two-dimensional case. Second, Squire's theorem is also usually interpreted as implying that if the two-dimensional problem is stable at some Reynolds number \( \tilde{R} \), then so is the three-dimensional problem at the same \( \tilde{R} \). This implication would be true if the neutral stability curve in the \( \alpha-\tilde{R} \) plane were to consist of one concave-up line, as is the usual case for one-fluid flows. However, in the two-layer flows, it is known that the neutral stability curve can consist of a concave-up line and a simple closed curve below. Also, it is possible that flows are unstable at zero Reynolds number, as in the case of adverse density stratification. In these cases, as pointed out in Ref. 31, the second portion of Squire's theorem would not hold. In the former situation, the first criticality is a point on the closed curve. If the point
\((\tilde{a}_2, \tilde{R}_2)\) is taken inside the island of instability in the \(\tilde{a}-\tilde{R}\) plane, and the line along \(\tilde{R} = \tilde{R}_1\) is just above this island but below the concave-up line, then there is linear stability in the two-dimensional problem for the Reynolds number \(\tilde{R}_1\), \(\tilde{R}_1 > \tilde{R}_2\). Now consider the three-dimensional problem with Reynolds number \(\tilde{R}_1\). Is it possible to satisfy a Squire's transformation \(\tilde{a}_2, \tilde{R}_2 = a_1, \tilde{R}_1\) for some choice of \(a\)? Indeed, this can be satisfied because the transformation between the wavenumbers stipulates only that \(a > \tilde{a}_2\), and since \(\tilde{R}_1 > \tilde{R}_2\), we can find such an \(a\). Thus, it is possible to choose the \(a\) and \(\beta\) in the three-dimensional problem at the Reynolds number \(\tilde{R}_1\), such that they correspond to the two-dimensional instability at \((\tilde{a}_2, \tilde{R}_2)\). Therefore the second portion of Squire's theorem is not always true for two-layer flows. An example is Fig. 5 in Ref. 9 for a combined Couette–Poiseuille flow simulating a flow of air over water. In Sec. 2.2 of Ref. 9, it is noted that a number of two-layer Poiseuille flows were checked for any islands of instability and none were found. To check whether the second portion of the theorem is true or not, one would have to compute the neutral stability curve in the \(\tilde{a}-\tilde{R}\) plane for each data set. This is not done in this paper. Instead, the neutral stability curves will be presented in the \(m-i_1\) plane.

Figure 1 displays neutral stability curves for Couette flow for a typical situation at low Reynolds number, with equal density for \(m < 1\). The dashed curve is derived from the long wave asymptotic formula and is the neutral stability curve for situations, in which instabilities from order unity and shorter wavelengths are suppressed by large amounts of surface tension. The bold curve is for \(T = 0.01\). As the surface tension increases, the curve moves toward that of the long waves. The region to the left of each curve is linearly stable and the region to the right is unstable. For small \(m\) and for \(m\) close to 1, the neutral stability curve for \(T = 0.01\) coincides with the long wave curve. This is explicitly illustrated in Fig. 2, which shows the critical wavenumbers for the case of \(T = 0.01\) from Fig. 1. Thus the critical wavenumbers for a large range of viscosity ratios lies in the order unity range.

The situation for the viscosity ratio \(m > 1\) is displayed in Figs. 3 and 4. In Fig. 3, the dashed curve is again derived from the long wave asymptotic formula. The bold curves are for \(T = 0.01\) and 0.1, showing the approach of the curve to the long wave one as the surface tension increases. Here, the region to the right of each curve is linearly stable and that to the left is unstable. As \(m\) increases in this graph, the bold curves tend toward \(i_1 = 1\), away from the long wave curve. This exemplifies the tendency for thin-layer arrangements to be linearly stable, as in the case of the core-annular pipe flow. Figure 4 illustrates that the critical wavenumber tends to infinity as \(m\) increases. For such short waves, one may set

![Figure 1](image1.png)

![Figure 2](image2.png)

![Figure 3](image3.png)

![Figure 4](image4.png)
Re $\sigma$ equal to 0 in Eq. (47) and find out the behavior of the critical wavenumber (also see Ref. 17).

Nonlinear calculations along the curves of Figs. 1–4 can be carried out provided there is a single finite critical wavelength. For example, a situation where long waves are unstable is not handled by the present theory. In such a case, the reader is referred to Refs. 5–7. Thus the range of viscosity ratios in which nonlinear calculations were carried out are 0.005 < $m$ < 0.727 and 1.3 < $m$ < 8.0. These calculations at several data points along the neutral stability curves yield supercritical bifurcations. The values of $\kappa$ for several situations are tabulated in Table I. In the column caption, FPG stands for "fixed pressure gradient" and refers to the approach where the pressure gradient in the $x$ direction is fixed in the nonlinear calculations and the equations for $\chi$ are (30)–(39). FVF stands for "fixed volume flux" and refers to the approach where the combined volume flux is fixed in the nonlinear calculations and the equations for $\chi$ are (34)–(39) and (41)–(45). The data shown have been checked for convergence. In Table I, both the fixed volume flux approach and the fixed pressure gradient approach are quantitatively similar. This need not be the case when the parameters are more extreme; for example, see the results for large Reynolds number and large $m$ in the Appendix.

The introduction of a small adverse density stratification on the arrangement in Fig. 3 at $T = 0.1$ is shown in Fig. 5. The main difference here is that the long wave curve approaches $l_i = 1$ as $m$ approaches 1. This is because at $m = 1$, the adverse density stratification yields long wave instability for all arrangements with a heavier fluid above ($l_i < 1$). Comparing with Fig. 3, the stable region is decreased slightly, as expected, but the overall difference is slight. Nonlinear results are obtained at several points along the neutral stability curve for $T = 0.1$, where $1.72 < m < 8.0$, i.e., the critical situation is not the long waves. Several of these data are presented in Table II. These yield supercritical bifurcations so that waves are supported, even in the presence of an adverse density stratification. A more pronounced density stratification is $r = 0.25$, $1/F^2 = 10.0$, with the other parameters as in Fig. 3, and $T = 0.1$. This situation is shown in Fig. 6. Nonlinear computations at several points along the neutral stability curve for the range of viscosity ratio $1.5 < m < 7.0$ again yield supercritical bifurcations.

Computations for a Poiseuille flow were performed for the equal density case, as shown in Fig. 7: $\mu_1 = 1$, $\mu_2 = 1$, $l^* = 5$, $G^* = 2$, $\mu_1 = 1/m$, and $m > 1$. It is more appropriate to keep these dimensional quantities fixed rather than to keep the dimensionless parameters fixed, for the calculation of neutral stability curves in the $m$-$l_i$ plane. This is because for a Poiseuille flow, the interfacial speed $U_i$ → 0 as $l_i → 1$ and the dimensionless parameters $G$ and $T$ contain $U_i$ in their denominators. Thus keeping $G$ and $T$ fixed in the $m$-$l_i$ plane would not reflect the parameter variations that would be analogous to an experiment, e.g., the stability of the thin-layer arrangement ($l_i$ close to 1) for large $m$ would not show up. Therefore, with regard to experiments, it is more natural in this case to keep the above quantities fixed and to vary $\mu_2$.

The neutral stability curves in Fig. 7 are similar to those of the Couette flow of Fig. 3. There is a difference in the angle that the curve for surface tension 0.05 makes at the junction with the long wave curve. In the Poiseuille flow, there are

<p>| Table I. Nonlinear results for Couette flow, $R_i = 10$, $G = 0$, $r = 1$, $1/F^2 = 0$, and $T = 0.01$. |
|----|----|----|----|----|----|----|</p>
<table>
<thead>
<tr>
<th>$m$</th>
<th>$l_i$</th>
<th>$\alpha$</th>
<th>$\Re \sigma$</th>
<th>$\kappa_{\text{FPG}}$</th>
<th>$\kappa_{\text{FVF}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.23</td>
<td>1.0</td>
<td>0.39E - 4</td>
<td>-0.487324</td>
<td>-0.463502</td>
</tr>
<tr>
<td>0.05</td>
<td>0.215</td>
<td>4.5</td>
<td>0.14E - 5</td>
<td>-0.51299</td>
<td>-0.4311273</td>
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<tr>
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<td>0.191</td>
<td>9.4</td>
<td>-0.19E - 3</td>
<td>-0.6871509</td>
<td>-0.6741274</td>
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<tr>
<td>0.5</td>
<td>0.372</td>
<td>6.3</td>
<td>-0.16E - 4</td>
<td>-0.156980</td>
<td>-0.157963</td>
</tr>
<tr>
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<td>0.495</td>
<td>2.3</td>
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<td>-0.398115</td>
<td>-0.400114</td>
</tr>
<tr>
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<td>6.552</td>
<td>4.0</td>
<td>0.35E - 4</td>
<td>-0.373 - 0.511</td>
<td>37.3 - 528</td>
</tr>
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<td>0.951</td>
<td>30.0</td>
<td>-0.66E - 3</td>
<td>-0.10915</td>
<td>-0.10915</td>
</tr>
<tr>
<td>8.0</td>
<td>0.966</td>
<td>39.0</td>
<td>-0.38E - 3</td>
<td>-0.284</td>
<td>-0.37960</td>
</tr>
</tbody>
</table>

FIG. 5. Neutral stability curves for $R_i = 10$, $G = 0$, $r = 0.95$, $1/F^2 = 1.0$, and $m > 1$, in the $l_i$-$m$ plane. The dashed curve is for the long wave case. The bold curve is the case $T = 0.1$. The region to the left of each curve represents instability, and the region to the right represents linear stability.

<p>| Table II. Nonlinear results for Couette flow, $R_i = 10$, $G = 0$, $r = 0.95$, $1/F^2 = 1$, and $T = 0.1$. |
|----|----|----|----|----|----|----|</p>
<table>
<thead>
<tr>
<th>$m$</th>
<th>$l_i$</th>
<th>$\alpha$</th>
<th>$\Re \sigma$</th>
<th>$\kappa_{\text{FPG}}$</th>
<th>$\kappa_{\text{FVF}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.72</td>
<td>0.59</td>
<td>2.1</td>
<td>-0.23E - 3</td>
<td>-8.1</td>
<td>-8.2</td>
</tr>
<tr>
<td>2.0</td>
<td>0.621</td>
<td>4.3</td>
<td>0.47E - 3</td>
<td>-4.3</td>
<td>-272</td>
</tr>
<tr>
<td>3.0</td>
<td>0.767</td>
<td>7.9</td>
<td>0.72E - 3</td>
<td>-173</td>
<td>-935</td>
</tr>
<tr>
<td>4.0</td>
<td>0.84</td>
<td>10.2</td>
<td>-0.68E - 3</td>
<td>-166</td>
<td>-1582</td>
</tr>
<tr>
<td>5.0</td>
<td>0.879</td>
<td>12.3</td>
<td>-0.46E - 3</td>
<td>-175</td>
<td>-2501</td>
</tr>
<tr>
<td>8.0</td>
<td>0.932</td>
<td>18.5</td>
<td>-0.53E - 3</td>
<td>-277</td>
<td>-7414</td>
</tr>
</tbody>
</table>
TABLE III. Critical conditions for the PPF profile.

<table>
<thead>
<tr>
<th>( R_1 )</th>
<th>( \alpha )</th>
<th>( T )</th>
<th>( 1/F^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2983.0065</td>
<td>3.6354</td>
<td>24379.048</td>
<td>6.8033538</td>
</tr>
<tr>
<td>3490.6171</td>
<td>5.0852</td>
<td>41667.624</td>
<td>39.748162</td>
</tr>
<tr>
<td>6190.3348</td>
<td>7.5008</td>
<td>46991.229</td>
<td>101.10745</td>
</tr>
</tbody>
</table>

TABLE IV. Critical conditions for the BL1 profile.

<table>
<thead>
<tr>
<th>( G )</th>
<th>( R_1 )</th>
<th>( \alpha )</th>
<th>( T )</th>
<th>( 1/F^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1504E-2</td>
<td>1837.6977</td>
<td>2.8322</td>
<td>39.572811</td>
<td>17.926</td>
</tr>
<tr>
<td>0.6614E-3</td>
<td>4179.0355</td>
<td>3.5002</td>
<td>34.83066</td>
<td>27.73124</td>
</tr>
<tr>
<td>0.7767E-3</td>
<td>3558.7535</td>
<td>3.9178</td>
<td>81.739704</td>
<td>305.92551</td>
</tr>
<tr>
<td>0.3807E-3</td>
<td>7259.5709</td>
<td>4.5</td>
<td>80.140125</td>
<td>588.13848</td>
</tr>
</tbody>
</table>

TABLE V. Critical conditions for the BL2 profile.

<table>
<thead>
<tr>
<th>( G )</th>
<th>( R_1 )</th>
<th>( \alpha )</th>
<th>( T )</th>
<th>( 1/F^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5453E-3</td>
<td>4124.2369</td>
<td>2.6239</td>
<td>26449.572</td>
<td>12.012078</td>
</tr>
<tr>
<td>0.5008E-3</td>
<td>4491.3724</td>
<td>4.0956</td>
<td>48575.041</td>
<td>81.028383</td>
</tr>
<tr>
<td>0.2182E-3</td>
<td>10321.35</td>
<td>4.6311</td>
<td>42312.101</td>
<td>122.96174</td>
</tr>
</tbody>
</table>

TABLE VI. Critical conditions for the PCF profile.

<table>
<thead>
<tr>
<th>( R_1 )</th>
<th>( \alpha )</th>
<th>( T )</th>
<th>( 1/F^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1292.1093</td>
<td>3.0004</td>
<td>56282.285</td>
<td>36260.272</td>
</tr>
<tr>
<td>8594.5979</td>
<td>4.562</td>
<td>33845.847</td>
<td>52451.793</td>
</tr>
</tbody>
</table>

TABLE VII. Nonlinear results for the PPF profile.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \kappa_{(FPG)} )</th>
<th>( \kappa_{(FVF)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.124E-4</td>
<td>-5.648</td>
<td>-5.6256</td>
</tr>
<tr>
<td>0.9905E-5</td>
<td>-19.3944</td>
<td>-0.4948E-1</td>
</tr>
<tr>
<td>-0.184E-5</td>
<td>-35.282</td>
<td>-0.3488</td>
</tr>
</tbody>
</table>

TABLE VIII. Nonlinear results for the BL1 profile.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \kappa_{(FPG)} )</th>
<th>( \kappa_{(FVF)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2E-2</td>
<td>-9.332</td>
<td>-0.27</td>
</tr>
<tr>
<td>-0.1E-2</td>
<td>-12.601</td>
<td>-0.49</td>
</tr>
<tr>
<td>-0.2E-2</td>
<td>-37.374</td>
<td>-0.37</td>
</tr>
<tr>
<td>-0.2E-2</td>
<td>-54.720</td>
<td>-0.11</td>
</tr>
</tbody>
</table>

TABLE IX. Nonlinear results for the BL2 profile.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \kappa_{(FPG)} )</th>
<th>( \kappa_{(FVF)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3E-5</td>
<td>-7.740</td>
<td>-0.395</td>
</tr>
<tr>
<td>0.5E-5</td>
<td>22.046</td>
<td>-0.23</td>
</tr>
<tr>
<td>-0.8E-5</td>
<td>28.077</td>
<td>-0.75</td>
</tr>
</tbody>
</table>

situations where there is linear stability for two disjoint intervals of \( l_1 \), at fixed \( m \); e.g., at \( m = 1.2 \), stability for \( 0.53 < l_1 < 0.58 \) and \( l_1 > 0.69 \). In particular, at the junction, around \( m = 1.7 \), \( l_1 = 0.58 \), there is a situation with both a long wave criticality and a criticality at a single nonlong wave value of the wavenumber. This junction represents a crossover of the two neutral stability curves. For large \( m \), there is the familiar stability of thin layers; e.g., at \( m = 5 \), stability for \( l_1 > 0.97 \). As in the Couette case of Fig. 3, the critical wavenumber \( \alpha \), increases as \( m \) increases. Nonlinear calculations at several points along the bold curve show supercritical bifurcations.

APPENDIX: COMPARISON WITH MODULATED WAVE SOLUTIONS

Blennerhasset follows the ideas of Ref. 21 and analyzed the nonlinear amplitude equation for two-layer Couette–Poiseuille flow. Some differences between his approach and the present one have been mentioned in Sec. III. It is of interest to compare his results with those of the present approach. This appendix presents numerical results for some of the situations tabulated in his Tables 2–5, 7–10. Here Tables III–VI correspond to his Tables 2 (the first three entries), 3a (the first four entries), 4a (the first three entries), and 5 (the first two entries), respectively. The corresponding nonlinear results are in Tables VII–X. These results require less than 70 Chebyshev modes in each fluid for convergence. Those data sets in the tables of Ref. 9 with higher Reynolds numbers require more Chebyshev modes and are not pursued here.

In all situations, fluid 1 is water and fluid 2 is air. The viscosity and density ratios are \( \mu = 64.009897 \), \( r = 815.59184 \). Table III is his PPF (two-layer plane Poiseuille flow) profile with \( l_1 = 0.5 \). Table IV is his BL1 (boundary layer 1) profile with \( l_1 = 0.5 \). Table V is his BL2 profile with \( l_1 = 3 \) and Table VI is his PCF (two-layer plane
TABLE X. Nonlinear results for the PCF profile.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\kappa$(FPG)</th>
<th>$\kappa$(FVF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.44E - 4 - 12.849i</td>
<td>- 0.14 - 0.32i</td>
<td>0.135 - 20.46i</td>
</tr>
<tr>
<td>0.116E - 4 - 19.128i</td>
<td>- 0.4974 + 6.402i</td>
<td>- 0.3852 - 20.955i</td>
</tr>
</tbody>
</table>

Couette flow) profile with $\lambda = 0.5$. Tables VII–X present the corresponding critical eigenvalues and the $\kappa$ of (40). In the column caption, FPG stands for "fixed pressure gradient" and refers to the approach where the pressure gradient in the $x$ direction is fixed in the nonlinear calculations and the equations for $\chi$ are (30)–(39). FVF stands for "fixed volume flux" and refers to the approach where the combined volume flux is fixed in the nonlinear calculations and the equations for $\chi$ are (34)–(39) and (41)–(45). The data shown have been checked for convergence.

The critical eigenvalues agree with those of Ref. 9. For example, his first BL2 profile in Blennerhassett’s Table 4a lists $c_x = 1.180$, which, in our notation, yields that the imaginary part of $\sigma$ should be $-7.7433$. Using 35 Chebyshev modes in each fluid, the computed result with quadruple precision on a Vax 11/785 was $\sigma = 0.699E - 6 - 7.7403i$.

Comparing the fixed volume flux results for $\kappa$ in Tables VII–X with those of Ref. 9, both approaches predict the same type of bifurcation, supercritical (Re $\kappa < 0$) or subcritical (Re $\kappa > 0$), except for the first entries in BL1 and PCF profiles. In Ref. 9, these entries are supercritical. Both approaches yield supercritical bifurcations for most of the cases displayed, and both yield subcritical bifurcations for the last entry in the PPF profile. The fixed pressure gradient approach yields supercritical bifurcations for all the cases displayed. Therefore these comparisons reveal some cases where the agreement is not even qualitative.

It is interesting that there are flow profiles where the bifurcation is supercritical for one formulation and subcritical for another (FPG or FVF formulation). This is obviously not intended to imply that the flow might be both supercritical and subcritical “at the same time”: the two formulations are two quite distinct problems.

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