Integral Notation

The notation for integrals is suggestive and easy to use but invites errors.

In definite integrals such as

\[ \int_a^b f(c, y) \, dy \]

the “\( dy \)” identifies \( y \) as the variable of integration. This is a dummy variable that gets “used up” in the integration process, so it is an *illegal expression* for this variable to occur outside the integrand (the boxed area). \( \int_a^b f(c, y) \, dy \) is the most common illegal expression, and usually indicates an error somewhere. It is always graded as wrong.

Indefinite integrals work differently: the notation \( g(c, y) = \int f(c, y) \, dy \) makes sense and is an alternate notation for \( \frac{d}{dy} g(c, y) = f(c, y) \). Here “\( dy \)” identifies the differentiation variable, and the final result is still a function of \( y \).

These differences lead to some differences in techniques. For instance when using substitution to transform an indefinite integral, a new variable is introduced and this information must be carried along. For example

\[ \int \cos(x^2) \, 2x \, dx = \int \cos(u) \, du, \text{ where } u = x^2 \]

The integral on the left is a function of \( x \), the one on the right is a function of \( u \), and the “where \( u = x^2 \)” is necessary to relate them. The equation is incomplete without this. This does not arise with definite integrals. See Substitution.

The form \( f(b) - f(a) \) occurs so frequently that there is a notation for it:

\[ f \bigg|_a^b = f(b) - f(a) \text{ or } f(x, c) \bigg|_{x=a}^{x=b} = f(b, c) - f(a, c) \]

The second form is used when \( f \) involves several symbols and it may not be clear which is being evaluated. Using this notation the relation between definite and indefinite integrals is written as:

\[ \int_a^b f(x) \, dx = \left( \int f \, dx \right) \bigg|_a^b \]