Completing the square

This is a method for eliminating the first-order term from quadratic polynomials, and is a key part of integration of quotients of polynomials. To simplify $ax^2 + bx + c$ first arrange the coefficient on the $x^2$ term to be a square: $\frac{1}{a}(a^2x^2 + abx + ac)$. Then find a linear polynomial whose square has the same $x$ and $x^2$ terms: $(ax + \frac{b}{2})^2 = a^2x^2 + abx + \frac{b^2}{4}$. Denote this linear polynomial by $z(x)$, then the quadratic becomes $\frac{1}{a}(z^2 + C)$, where $C = ac - \frac{b^2}{4}$.

- Setting this equal to zero and solving (first for $z$, then for $x$) gives the quadratic formula.
- This version of the procedure (and particularly the first step) is optimized to give the simplest exact arithmetic.

If the transformed constant term is positive we can write it as a square: $C = r^2$, otherwise as the negative of a square. The sign of the constant term determines the number of real roots of the quadratic. Putting these observations together gives the form we will use for integration: given $ax^2 + bx + c$ there is a linear $y(x)$ so that

- **no real roots**: the quadratic transforms to $\frac{1}{a}(y^2 + r^2)$
- **one real root**: the quadratic transforms to $\frac{1}{a}y^2$
- **two real roots**: the quadratic transforms to $\frac{1}{a}(y^2 - r^2)$

The are formulas $y = ax + \frac{b}{2}$ and $r = \sqrt{|ac - \frac{b^2}{4}|}$, but it seems to be easier and more reliable to go through the process than to try to remember the formulas.

**Example 1** Find a linear $y(x)$ and constant $r$ so that $3x^2 + 4x + 7 = \frac{1}{3}(y^2 + r^2)$

First complete the square to combine $x$ and $x^2$ terms:

$$\frac{1}{3}(((3x)^2 + 3(4x) + 3(7)) = \frac{1}{3}((3x + 2)^2 - (2)^2 + 21) = \frac{1}{3}((\cdots)^2 + 17)$$

Note the constant term is positive, indicating that there are no real roots, and we can write it as the square of its square root. Therefore

$$y(x) = 3x + 2, \quad r = \sqrt{17}$$
Example 2  Find $\int \frac{1}{3x^2+4x+7} \, dx$

The denominator has no real roots so we know this can be transformed by linear substitution to connect with the standard integral $\int \frac{1}{y^2+r^2} \, dy = \frac{1}{r} \tan^{-1}\left(\frac{y}{r}\right)$. Example 1 shows $y(x) = 3x + 2$ transforms the quadratic to $\frac{1}{3}(y^2 + (\sqrt{17})^2)$. The derivative is $y' = 3$ so doing the substitution introduces the inverse of this. The transformed integral is therefore

$$\int \frac{1}{\frac{1}{3}(y^2 + (\sqrt{17})^2)} \left(\frac{1}{3}\right) \, dy = \int \frac{1}{y^2 + (\sqrt{17})^2} \, dy$$

Using the standard integral and plugging in $y(x)$ gives

$$\int \frac{1}{3x^2+4x+7} \, dx = \frac{1}{\sqrt{17}} \tan^{-1}\left(\frac{3x+2}{\sqrt{17}}\right)$$

Example 3  Find $\int \frac{1}{\sqrt{-3x^2-4x+1}} \, dx$

The denominator has two real roots and the $x^2$ coefficient is negative so we know this can be transformed to connect with the standard integral $\int \frac{1}{\sqrt{r^2-y^2}} \, dy = \frac{1}{r} \sin^{-1}\left(\frac{y}{r}\right)$. Go through the procedure:

$$\frac{-1}{3}(3^2x^2 + 3(4x) - 3) = \frac{-1}{3}((3x - 2)^2 - 4 - 3) = \frac{-1}{3}((3x - 2)^2 - 7)$$

Take the $-1$ inside the parentheses and write $7 = (\sqrt{7})^2$, then this becomes $\frac{1}{3}((\sqrt{7})^2 - (3x - 2)^2)$ and the integral can be rewritten as

$$\int \frac{\sqrt{3}}{\sqrt{(\sqrt{7})^2 - (3x - 2)^2}} \, dx$$

Doing the linear substitution $y = 3x - 2$ introduces a factor $\frac{1}{3}$ so we get

$$\int \frac{\sqrt{3}}{\sqrt{(\sqrt{7})^2 - y^2}} \left(\frac{1}{3}\right) \, dy = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{(\sqrt{7})^2 - y^2}} \, dy = \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3x+2}{\sqrt{7}}\right)$$