**Change of Variables**

The definite integral form of the chain rule is:

$$\int_{a}^{b} f(u(x)) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du$$

This is called “substitution” when used from left to right, and “change of variables” when used from right to left. This note concerns change of variables; for substitution see the reference note Substitution.

Change of variables is easy to do: decide how you want the old variable to depend on the new one, and plug in. It is correspondingly less useful as a computational tool. In higher dimensions it is used to change coordinates, for instance from rectangular to polar. In one variable it is used to get access to functional identities, particularly trig identities, and in this case it is traditionally (but erroneously) called “trig substitution”.

**Example 1** Find $\int \sqrt{r^2 - x^2} \, dx$.

This is the area of a truncated circle, so is a natural problem. We would like to express $r^2 - x^2$ as a square so we can take the square root of it. Set $x = r \sin(t)$, and then

$$r^2 - x^2 = r^2 - r^2 \sin^2(t) = (r \cos(t))^2.$$  

This simplifies the factor we started with, but remember that the method introduces the derivative of the new function:

$$\int \sqrt{r^2 - x^2} \, dx = \int (r \cos(t)) x'(t) \, dt = \int (r \cos(t))^2 \, dt.$$  

This can be evaluated as $(r^2/2)(\sin(t) \cos(t) - t)$, for instance using integration by parts. Transforming back gives $\frac{1}{2}(x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r))$.

The following summarizes the forms encountered, identities to be used, and the change of variables:

<table>
<thead>
<tr>
<th>form</th>
<th>target identity</th>
<th>change of variables</th>
<th>outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + r^2$</td>
<td>$\tan^2 + 1 = \sec^2$</td>
<td>$x(t) = r \tan(t)$</td>
<td>$r^2 \sec^2(t)$</td>
</tr>
<tr>
<td>$r^2 - x^2 \geq 0$</td>
<td>$1 - \sin^2 = \cos^2$</td>
<td>$x(t) = r \sin(t)$</td>
<td>$r^2 \cos^2(t)$</td>
</tr>
<tr>
<td>$x^2 - r^2 \geq 0$</td>
<td>$\sec^2 - 1 = \tan^2$</td>
<td>$x(t) = r \sec(t)$</td>
<td>$r^2 \tan^2(t)$</td>
</tr>
</tbody>
</table>
Some integrals included in the standard table can now be derived: use $x = r \tan(t)$ to get

$$\int \frac{1}{x^2 + r^2} \, dx = \int \frac{1}{r^2 \sec^2(t)} (r \tan(t))' \, dt = \int \frac{1}{r} \, dt = \frac{t}{r}$$

solving gives $t = \arctan(\frac{x}{r})$, and the integral is $\frac{1}{r} \arctan(\frac{x}{r})$ as expected. The change of variables $x = r \sin(t)$ similarly gives the inverse sin integral.

**Example 2** Use trig change of variables to transform the following:

1. $\int \ln(r^2 + x^2) \, dx$
2. $\int y^3 \sqrt{5 - y^2} \, dy$
3. $\int \ln(x^2 - 9) \, dx$

$x = r \tan(t)$ takes the first to $\int \ln(r^2 \sec^2(t)) \, r \sec^2(t) \, dt$; $y = \sqrt{5} \sin(t)$ takes the second to $5^{5/2} \int \sin^3(t) \cos^2(t) \, dt$; and $x = 3 \sec(t)$ takes the third to $\int \ln(9 \tan^2(t)) \, 3 \sec(t) 3 \tan(t) \, dt$

**Caution:** All these examples can be better handled using other methods: parts for (1) and (3), substitution $u = 5 - y^2$ in (2). Check for this first, and use trig change of variables only as a last resort. This means the ones that must be done this way are usually tough.

**Example 3** Transform $\int \frac{1}{\sqrt{x^2 + r^2}} \, dx$ to a trig integral.

Since nothing else seems to work, try setting $x = r \tan(t)$. This transforms the integral to

$$\int \frac{1}{r \sec(t)} (r \sec^2(t)) \, dt = \int \sec(t) \, dt$$

There is a long evaluation of this using standard methods, not given here.