Test 2

• Covers *everything* we have done:
  – Lectures
  – In Class Assignments
  – Lab (but not Excel)

• Der 1090, 7pm Thursday (bring pencil, rubber, calculator)
Test 2 Review
Lesson 8: Integral Properties and Average Value

**Additive Property**
\[ \int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx \]

**Sums and Differences**
\[ \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \]

**Constant Multiples**
\[ \int_a^b c \, f(x) \, dx = c \int_a^b f(x) \, dx \]

**A-**
\[ \int_a^a f(x) \, dx = 0 \]

**B-**
\[ \int_b^a f(x) \, dx = -\int_a^b f(x) \, dx \]

**Average Value of a Function**
The average value of a function \( f(x) \) on an interval \([a, b]\) is
\[ \frac{1}{b-a} \int_a^b f(x) \, dx \]
Lesson 9: Population Growth

To model constant absolute growth, use $P(t) = A + B\ t$

Where: $A$ is the initial population and $B$ is the absolute growth rate

Given the absolute growth rate $P'(t)$, it is trivial to find the change in population from $t = a$ to $t = b$:

$$P(b) - P(a) = \int_a^b P'(t)\ dt$$

Start with population $A$, and have relative growth rate $r$.

How can we model this? $P(t) = A(1 + r)^t$

Found the relative change in the population based on its relative growth rate, by integrating the relative growth rate to find $\ln(P(b)/P(a))$, and then solving for $P(b)/P(a)$. 
Population Growth

Find a formula for $P(t)$ if $P(0) = 1,200$ and

- $P$ decreases by 12 per year $P(t) = 1200 - 12t$
- $P$ decreases by 1% per year $P(t) = 1200(1 - 0.01)^t$

A population is of size $P(t)$ after $t$ years. The relative rate of growth of $P(t)$ is $\frac{P'(t)}{P(t)}$. If we determine $\int_2^5 \frac{P'(t)}{P(t)} dt = -0.3$ how has the population changed from 2 to 5 years?

$$\int_2^5 \frac{P'(t)}{P(t)} dt = -0.3 = \ln \left| \frac{P(5)}{P(2)} \right| \Rightarrow \frac{P(5)}{P(2)} = e^{-0.3} = 0.741$$

$0.741 < 1.0 \therefore$ population changed by $(0.741 - 1) = -0.259$

population decreased by $\approx 26\%$
Lesson 10: Newton’s Law of Cooling

The rate at which an object cools down or heats up…

That is,

\[ y' = k(y - A) \]

Where \( y(t) = \text{temperature at time } t, \) and

A is the ambient (surrounding) temperature.

The Temperature of the object after time \( t \) has passed is

\[ y = A + Ce^{kt} \]

\( y' = k(y - A) \): growth proportional to a difference.

Equilibrium solution: \( y = A \) General solution: \( y = A + Ce^{kt} \)
Newton’s Law of Cooling

1- Find the solution for: \( \frac{dP}{dt} = 5P - 50 \) and \( P(0) = 8 \)

\[
P' = 5(P - 10)P(t) = 10 + Ce^{5t} \iff P(t) = 10 - 2e^{5t}
\]

2- If an object takes 40 minutes to cool from 30 degrees to 24 degrees in a 20 degree room, how long will it take the object to cool to 21 degrees?

**Solve for t when \( y(t) = 21 \)**

\[
y(0) = 20 + Ce^{k*0} = 20 + C = 30 \iff \boxed{C = 10}
\]

\[
y(40) = 24 = 20 + 10e^{40k} \iff k = \frac{\ln(2/5)}{40} = \boxed{-0.023}
\]

\[
y(t) = 21 = 20 + 10\exp(-0.023 * t) \iff t = \frac{-40\ln(10)}{\ln(2/5)} = \boxed{100.5}
\]
**Lesson 11: Antiderivatives**

**Def:** We say $F$ is an *antiderivative* for $f$ if $F'(x) = f(x)$.

<table>
<thead>
<tr>
<th>If $f(x)$ is...</th>
<th>...then an antiderivative is...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{1}{n+1} x^{n+1}$ except if $n = -1$</td>
</tr>
<tr>
<td>$k$</td>
<td>$kx$ (assuming the variable is $x$!)</td>
</tr>
<tr>
<td>$\cos(kx)$</td>
<td>$\sin(kx)/k$</td>
</tr>
<tr>
<td>$\sin(kx)$</td>
<td>$-\cos(kx)/k$</td>
</tr>
<tr>
<td>$e^{kx}$</td>
<td>$e^{kx}/k$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$a^x$</td>
<td>$a^x/\ln(a)$</td>
</tr>
</tbody>
</table>

\[ \int f(x) \, dx \] represent *all possible* antiderivatives of $f(x)$. Called the *indefinite integral* of $f(x)$. 

Lesson 12: Integrals by Substitution

\[ \int f(g(x)) g'(x) \, dx \quad \text{Let } u = g(x). \]

\[ \int f(g(x)) g'(x) \, dx = \int f(u) g'(x) \, dx = \int f(u) \, du \]

1) Choose \( u \).

2) Calculate \( du \). \quad du = \frac{du}{dx} \, dx

3) Substitute \( u \).
   
   Arrange to have \( du \) in your integral also.
   
   (All \( x \)s and \( dx \)s must be replaced!)

4) Solve the new integral.

5) Substitute back in to get \( x \) again.
Lesson 13: Solving Definite Integrals

Theorem: (Fundamental Theorem I)

Or: If $F$ is an antiderivative for $f$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

We have to

- find an antiderivative;
- evaluate at $b$;
- evaluate at $a$;
- subtract the results.

- We can use substitution
- Solve an indefinite integral first
- Change the limits

We have three methods:

1. Basic formulas
2. Algebraic simplification
3. Substitution
An Integral Problem

\[ \int_0^{\sqrt{\frac{\pi}{2}}} 5x \cos(5x^2) \, dx \]

Let \( u = 5x^2 \).
\( du = 10x \, dx \)

If \( x = 0 \), \( u = 5(0)^2 = 0 \)

If \( x = \sqrt{\frac{\pi}{2}} \), \( u = \frac{5\pi}{2} \)

\[ \int_0^{\sqrt{\frac{\pi}{2}}} 5x \cos(5x^2) \, dx = \frac{1}{2} \int_0^{\sqrt{\frac{\pi}{2}}} \cos(5x^2) \cdot 10x \, dx \]

\[ = \frac{1}{2} \int_0^{\frac{5\pi}{2}} \cos(u) \, du \]

\[ = \frac{1}{2} \sin(u) \bigg|_0^{\frac{5\pi}{2}} \]

\[ = \frac{1}{2} \sin\left(\frac{5\pi}{2}\right) - \frac{1}{2} \sin(0) \]

\[ = \frac{1}{2} \]
Lesson 14: The Improper Integral

Integrals with one or both limits infinite are referred to as *improper* integrals.

\[ \int_{a}^{\infty} f(x) \, dx \quad \int_{-\infty}^{a} f(x) \, dx \quad \int_{-\infty}^{\infty} f(x) \, dx \]

are all considered improper.

With the preceding ideas, we give meaning to integrals with infinite limits:

\[ \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \]

will mean

\[ \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \]
Lesson 14: The Improper Integral

\[ \int_1^\infty e^{-2x} \, dx = \lim_{b \to \infty} \int_1^b e^{-2x} \, dx = \lim_{b \to \infty} \left. \frac{e^{-2x}}{-2} \right|_1^b = -\frac{1}{2} \left( \frac{1}{e^{2b}} - \frac{1}{e^{2(1)}} \right) \]

\[ = -\frac{1}{2} \left( \left( \lim_{b \to \infty} \frac{1}{e^{2b}} \right) - \frac{1}{e^{2(1)}} \right) = -\frac{1}{2} \left( 0 - \frac{1}{e^2} \right) = \frac{1}{2e^2} \]

\[ \int_1^\infty \frac{1}{t} \, dt = \lim_{b \to \infty} \int_1^b \frac{1}{t} \, dt = \lim_{b \to \infty} (\ln |t| \big|_1^b) \]

\[ = \lim_{b \to \infty} (\ln b - \ln 1) = \lim_{b \to \infty} \ln b = \infty (diverges!) \]
Lesson 15: Analyzing Antiderivatives

Finding $F$ from $F'$

$$F(b) = F(a) + \int_a^b F'(x) \, dx$$

Given the graph of $F'$ and $F(0)$, find the following:

$F(0)$, $F(1)$, $F(2)$, $F(3)$, $F(4)$, $F(5)$

g increases where the derivative is + and decreases where it is -

Local max, Local min:

Largest value and Smallest value
Antiderivatives

Locate the $x$ and $y$ coordinates of any local mins/maxes for $f$, given $f'(x)$ graphed at right, and the fact that $f(0) = -10$.

Local max: $x = 2$

$y = f(0) + \int_{0}^{2} f'(x) \, dx = -10 + 12 = 2$

Local min: $x = 5$

$y = f(2) + \int_{2}^{5} f'(x) \, dx = 2 - 16 = -14$