1- What is the average value of $f(x) = 7$ on the interval $[2, 6]$?

$$\frac{1}{b-a} \int_{a}^{b} f(x) \, dx = \frac{1}{6-2} \int_{2}^{6} 7 \, dx$$

$$= \frac{1}{4} (7 \times 4) = 7$$
2- Suppose the following integrals are known:

\[
\int_a^b f(x) \, dx = 4 \quad \int_a^b g(x) \, dx = -2
\]

\[
\int_a^b (f(x))^2 \, dx = 5 \quad \int_a^b (g(x))^2 \, dx = 7
\]

Find the integrals

\[
\int_a^a f(x) g(x) \, dx = 0
\]

\[
\int_a^b f(x) - 3g(x) \, dx = 10
\]

\[
\int_b^a -f(x) + 3g(x) \, dx = 10
\]

\[
\int_a^b (f(x))^2 \, dx - 2 \left( \int_a^b g(x) \, dx \right)^2 = -3
\]
Absolute Growth

Today: Several different ways to model the growth of populations.

Let $P(t)$ be the population at time $t$.

The *absolute growth rate* is the rate of change of $P$, or $P'(t)$.

Approximate with average absolute growth rate from $t = a$ to $t = b$:

$$\frac{P(b) - P(a)}{b - a}$$

Much like finding average velocity.
Example

19 squirrels in 1998
27 squirrels in 2000

What is the absolute growth rate?

\[
\frac{27 - 19}{2000 - 1998} = \frac{8}{2} = 4 \text{ squirrels / year}
\]

on average, the squirrel population went up about 4 squirrels per year in this time period.
Constant Absolute Growth

To model constant absolute growth, use \( P(t) = A + B t \)

Where:  
- \( A \) is the initial population  
- \( B \) is the absolute growth rate

**Example**

We start an exclusive club with 70 members, and recruit 7 new members per week.

After \( t \) weeks, we have

\[
P(t) = 70 + 7t 
\]

members.

Note:  
- \( P(0) = 70 \)  
- \( P(1) = 77 \)  
- \( P(2) = 84 \), etc.
Change in Population?

Given the absolute growth rate $P'(t)$, it is trivial (using FTC) to find the change in population from $t = a$ to $t = b$:

$$P(b) - P(a) = \int_{a}^{b} P'(t) \, dt$$
Relative Growth Rate

Growth as a *percentage* of current population, so relative growth rate is $P'(t)/P(t)$.

Average relative growth rate:

\[
\frac{P(b) - P(a)}{(b - a)P(a)}
\]

Often used to describe natural populations.
Example

19 squirrels in 1998
27 squirrels in 2000

What is the *relative* growth rate?

\[
\frac{27 - 19}{(2000 - 1998)(19)} = \frac{8}{38} \approx 0.21
\]

Or about 21% per year.
Constant Relative Growth

Start with population $A$, and have relative growth rate $r$.

How can we model this?

$P(1) = A + Ar = A(1 + r)$

$P(2) = A(1 + r) + [A(1 + r)]r = [A(1 + r)](1 + r) = A(1 + r)^2$

The pattern repeats… after $t$ years we have

$P(t) = A(1 + r)^t$

Like multiplying by “100 and $r$” percent each time
Example

Population starts at 1.7 million in 1970, and increases by 2% each year. 

After $t$ years, population is

$$P(t) = 1.7(1 + .02)^t = 1.7(1.02)^t \text{ million.}$$

Check: $P(1) = 1.7(1.02) = 1.734$ million.

Relative change:

$$\frac{1.734 - 1.7}{(1971 - 1970)(1.7)} = 0.02$$
Change in Population

For **absolute** growth, it is easy to get from growth rate $P'(t)$ to total absolute growth:

$$P(b) - P(a) = \int_a^b P'(t) \, dt$$

But the relative growth rate is $\frac{P'(t)}{P(t)}$!

What can we get from this?

**Question**

Find the derivative:

$$\frac{d}{dt} \ln(P(t)) = \frac{1}{P(t)} P'(t) = \frac{P'(t)}{P(t)}$$

So the relative rate of growth is the *derivative* of $\ln(P(t))$!
Relative Change

So by the fundamental theorem,

$$\int_a^b \frac{P'(t)}{P(t)} dt = \ln(P(b)) - \ln(P(a))$$

$$= \ln\left(\frac{P(b)}{P(a)}\right)$$

This would be the percentage change in the population!

To get $P(b)/P(a)$ out, we will have to raise $e$ to the power of the integral’s result.
Relative Change

To find relative change from a relative rate of change:

(1) Integrate the relative rate of change

(2) Raise $e$ to the power of the result.

(3) You now have $P(b)/P(a)$. 
Example

Estimate the relative change in the population from 1907 to 1911.

<table>
<thead>
<tr>
<th>Year</th>
<th>1907</th>
<th>1909</th>
<th>1911</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Growth Rate</td>
<td>0.01</td>
<td>0.005</td>
<td>0.021</td>
</tr>
</tbody>
</table>

Step 1: Integrate

\[
\int_{a}^{b} \frac{P'(t)}{P(t)}\,dt = \int_{1907}^{1911} \frac{P'(t)}{P(t)}\,dt
\]

\[
\approx \frac{2}{3} \left(0.01 + 4(0.005) + 0.021\right) = 0.034
\]

Step 2: Raise e to 0.034

\[e^{0.034} \approx 1.03458\]

So \(P(1911)\) is about 103.5% of \(P(1907)\).

Or: The population increased by about 3.5%.
Average vs. Instantaneous

Two ways to view “increase by 2% per year”:

- **Instantaneous rate of 2% per year**
  - This leads to $y' = 0.02 \ y$, and $y = Ce^{0.02t}$.

- **By the end of each year, we have increased by 2%**.
  - This leads to $A(1.02)^t$.

Note that:

If $y = Ae^{-02t}$, $y(1) = Ae^{02}$, or about 1.020201A.

If $y = A(1.02)^t$, then $y(1) = 1.02A$. 
Summary

• Average absolute \( \left( \frac{P(b) - P(a)}{b - a} \right) \) and relative \( \left( \frac{P(b) - P(a)}{(b - a)P(a)} \right) \) growth rates.

• Formulas for a population at time \( t \) if the population has a constant absolute ( \( P(t) = A + Bt \) ) or relative ( \( P(t) = A(1 + r)^t \) ) growth rate.

• Found the relative change in the population based on its relative growth rate, by integrating the relative growth rate to find \( \ln(P(b)/P(a)) \), and then solving for \( P(b)/P(a) \).

• Discussed the difference between constant instantaneous and constant average relative growth.
In-class Group Work

Relative growth rate given by the graph.

Find the relative change in the population from year 0 to 15.

Hint: Use what we know about the relationship between area and the integral.
In-class Group Work

Relative growth rate given by the graph.

Find the relative change in the population from year 0 to 15.

\[
\frac{P(15)}{P(0)} = e^{\int_0^{15} \frac{P'(t)}{P(t)} dt} = e^{0.1-0.025} = e^{0.075} \approx 1.078
\]

So \( P(15) \) is about 107.8% of \( P(0) \).

Or: The population increased by about 7.8%.