#1: State the fundamental theorem of calculus version I or II.

Version I:

\[ \int_{a}^{b} F'(t) \, dt = F(b) - F(a) \]

Version II:

\[ F(x) = \int_{a}^{x} f(t) \, dt \quad \text{and} \quad F'(x) = f(x) \]
Comments of last quiz

#1: State the fundamental theorem of calculus version I or II.

Version I.: 
If \( F'(t) \) is continuous for \( a \leq t \leq b \) then

\[ \int_{a}^{b} F'(t) \, dt = F(b) - F(a) \]

Version II.
If \( f \) is continuous on an interval, \( a \) is a fixed point in that interval, and \( F(x) \) is defined as

\[ F(x) = \int_{a}^{x} f(t) \, dt \]

, then \( F'(x) = f(x) \).
Comments of last quiz

The pace of this course is

Too fast: 2.5
Too slow: 2
Ok so far: 23.5
Exponential Growth and Decay
Review

Differential Equations: An equation with derivatives in it. They express the relationship involving the rates of change

- **Easy to confirm solutions**

1. Which of the following is a solution to the differential equation $y' + y = 0$?

   - $y = e^{2x}$
   - $y = x^2$
   - $y = e^{-x}$
   - $y = 2e^x$

   - **Is $y = e^{2x}$ a solution to $y' + y = 0$?** $y' + y = 2e^{2x} + e^{2x} = 3e^{2x} \neq 0$
     So **NO**, this is not a solution to the differential equation.

   - **Is $y = x^2$ a solution to $y' + y = 0$?** **NO, since** $y' + y = 2x + x^2 \neq 0$

   - **Is $y = e^{-x}$ a solution to $y' + y = 0$?** $y' + y = -e^{-x} + e^{-x} = 0$
     So **Yes**, this is a solution to the differential equation.

   - **Is $y = 2e^x$ a solution to $y' + y = 0$?** **NO, since** $y' + y = 2e^x + 2e^x = 4e^x \neq 0$
Review

• Harder to *find* solutions

We approximate solutions at a series of points, using Euler’s method.

\[ y(x + h) \approx y(x) + h \, y'(x) \]

For this class, we will usually use \( h = 1 \): \[ y(x + 1) \approx y(x) + y'(x) \]

2. Consider the initial value problem \( y' = 2y - t, \quad y(1) = 3 \).

Estimate \( y(2) \) and \( y(3) \).

\[ y(2) \approx y(1) + y'(1) = 3 + 2(3) - 1 = 8 \]

\[ y(3) \approx y(2) + y'(2) = 8 + 2(8) - 2 = 22 \]
Proportional Growth: $y' = ky$

- Rate of growth proportional to the function
- Large values of $y$ mean fast growth
- Small values of $y$ mean slow growth
Proportional Growth: \( y' = ky \)

**Applications**

1. **Balance of a bank account**: The interest paid is proportional to the amount present.

2. **Decay of a radioactive substance**: In this case, one-half of any amount present will decay in some fixed amount of time called a half-life.

3. **Amount of a drug absorbed**: The body will absorb a proportion of the drug present.

4. **The size of a population**: The larger a population, the more rapidly the population will grow. Again, the growth rate will be proportional to the size of the current population.
Solving the Equation

Special case: \( y' = y \).

One solution: \( y = e^x \).\[ \frac{d}{dx} e^x = e^x \]

Also have: \( y = Ce^x \).\[ \frac{d}{dx} Ce^x = Ce^x \]

In fact, all solutions are of the form \( \frac{d}{dx} Ce^x = Ce^x \) where \( C = y(0) \).

Now: \( y' = ky \)

Based on what we saw last class, try \( y = Ce^{kx} \):

\[ \frac{d}{dx} Ce^{kx} = k(Ce^{kx}) \] This works!
Solving the Equation

We say that $y = Ce^{kx}$ is the general solution to $y' = k \cdot y$.

(Note: we have not proved this!)

We call it a general solution because we do not know the value of $C$. If we also determine the constant $C$, we call the resulting solution the particular solution. For this, we need additional information: must have an initial value problem.
Example

Solve the initial value problem \( y' = 2y \) and \( y(0) = 3 \).

General solution: \( y = Ce^{2x} \)

If \( y(0) \) is to be 3, then \( y(0) = Ce^0 = 3 \), so \( C = 3 \).

The solution to this initial value problem is \( y = 3e^{2x} \).

Note About \( C \)

- If we are given \( y(0) \) as our initial value, then the \( C \) in \( y = Ce^{2x} \) is \( y(0) \).

- If we are given a different initial value, we will have to set up and solve an equation to find \( C \).
Example

Solve the initial value problem \( y' = -y \), \( y(2) = 1 \).

General Solution: \( y = Ce^{-x} \) \( (k = -1) \)

Since \( y(2) \) is to be 1, we must solve

\[
1 = Ce^{-2}
\]

to get

\[
C = e^2.
\]

Solution: \( y = e^2e^{-x} = e^{2-x} \)
Exponential Growth

\[ y' = k \cdot y, \quad k > 0: \text{Exponential growth} \]

Ex: \[ y' = 2y; \] Two solutions:

\[ y = 5e^{2x} \quad y = -e^{2x} \]

Exponential Decay

\[ y' = k \cdot y, \quad k < 0: \text{Exponential decay} \]

Ex: \[ y' = -y; \] Two solutions:

\[ y = 2e^{-x} \quad y = -3e^{-x} \]
Growth: Example

**Accruing Interest**

2003: Deposit $1,500 into a bank account. Interest is 7% compounded continuously. How much in 2013?

Let \( M(t) \) = amount after \( t \) years

\[
\frac{dM}{dt} = 0.07M
\]

(General Solution) \( M = Ce^{0.07t} \)

\( C = 1,500 \)

\( M = 1500e^{0.07t} \)

2013 corresponds to \( t = 10 \), so

\[
M(10) = 1500e^{0.07(10)} \approx 3,020.63
\]

Example

In 2003: Instead, deposit $2,000 at 9%. How much in 2008?

\[
M' = 0.09M
\]

\[
M(t) = 2000e^{0.09t}
\]

\[
M(5) = 2000e^{0.09(5)} \approx 3,136.62
\]
Growth: Example

Algae Population Growth

Given unlimited resources and space, populations grow at a rate proportional to the current size.

Algae population: Doubles every 7 hours. Start with 100 algae. How long will it take to grow 1,000,000?

Exponential growth: \( P(t) = P_0 e^{kt} \)

Initially, \( P(0) = 100 \), so we have

Find \( k \):

\[
P(7) = 100e^{7k} = 200
\]

\[e^{7k} = 2\]

\[7k = \ln(2)\]

\[k = \frac{\ln(2)}{7}\]

Population at \( t \):

\[P(t) = 100e^{\frac{\ln(2)}{7}t}\]
Growth: Example

Algae Population Growth

Algae population: Doubles every 7 hours. Start with 100 algae. How long will it take to grow 1,000,000?

Population at time $t$: $P(t) = 100e^{\frac{\ln(2)}{7}t}$

$100e^{\frac{\ln(2)}{7}t} = 1,000,000$

$e^{\frac{\ln(2)}{7}t} = 10,000$

$t = \frac{7\ln(10,000)}{\ln(2)} \approx 93$ hours
Growth: Example

Algae Population Growth

Algae population: What if it doubles every 10 hours, and we start with 20. How long will it take to grow 1,000,000?

Population: \( P(t) = 20e^{kt} \)

\( k: \)

\[ 40 = 20e^{k \cdot 10}, \text{ so } k = \frac{\ln(2)}{10} \]

\[ P(t) = 20e^{\frac{\ln(2)}{10}t} = 1,000,000 \]

\[ e^{\frac{\ln(2)}{10}t} = 50,000 \]

\[ t = \frac{10\ln(50,000)}{\ln(2)} \approx 156 \text{ hours} \]
Decay: Example

**Carbon Dating**

C-14: **Half-life approximately 5,730 years.**

Fossil with only 1% C-14 left. How old is the fossil?

All half-life problems: If y is percentage left, then

\[
\frac{dy}{dx} = ky
\]

Since \( y(0) = 1 \) (i.e., 100%), we have \( y(t) = e^{kt} \)

Find \( k \):

\[
y(5,730) = e^{5730k} = 0.5
\]

\[
5730k = \ln(0.5)
\]

\[
k = \ln(0.5) / 5730
\]

\[
\approx -0.000120968
\]

Age:

\[
0.01 = e^{-0.000120978 t}
\]

\[
t = \frac{\ln(0.01)}{-0.000120978} \approx 38,069 \text{ years}
\]
Decay: Example

**Carbon Dating**

C-14: Half-life approximately 5,730 years.

What if 20% of the C-14 remained?

Same $k$!

\[ 0.2 = e^{-0.000120968t} \]

\[ t = \frac{\ln(0.2)}{-0.000120968} \approx 13,305 \text{ years} \]
Decay: Example

**Chemical Dilution**

Tank: 90 gallons water and 10 gallons chemical. Water flows in, mixture flows out at 20 gal/min.

How many minutes to reduce concentration to 1%?

Let \( y(t) \): concentration at \( t \). Then

\[
y(0) = \frac{\text{vol of chem}}{\text{vol of tank}} = \frac{10}{100} = 0.1
\]

What’s \( y' \)? Rate of chemical flowing out is

(outflow rate)(concentration) = 20 \( y(t) \)

Rate of change of concentration?

\[
y' = -\frac{20y}{100} = -0.2y
\]

Solve: \( y(0) = 0.1 \), \( y' = -0.2y \). \( y(t) = 0.1e^{-0.2t} \)
Decay: Example

Chemical Dilution

Tank: 90 gallons water and 10 gallons chemical.

Water flows in, mixture flows out at 20 gal/min.

How many minutes to reduce concentration to 1%?

\[ 0.1e^{-0.2t} = 0.01 \]

\[ e^{-0.2t} = 0.1 \]

\[ t = \frac{\ln(0.1)}{-0.2} \approx 11.5 \text{ minutes} \]
Group work

Solve the initial value problem:

\[ y' + 2y = 0 \quad \text{and} \quad y(1) = 3 \]