Lesson 6

Differential Equations
Algebraic Equations:

Require a number that makes the equation true.

Ex: $x = 2$ and $x = -2$ are both solutions to $x^2 - 4 = 0$.

Differential Equations:

What is a differential equation?

An equation with derivatives in it.
Example: Population Growth

The rate at which a population grows is directly proportional to the size of the population itself.

\[ \frac{d}{dt} P = k P \]
Differential Equations:

Def: Require a *function* that makes the equation true. The equation uses the function and its derivative. (Often write the unknown function as $y$.)

- Many of the fundamental laws of physics, chemistry, biology and economics can be formulated as differential equations.

- They express the relationship involving the rates of change

- A solution to a differential equation is a function whose *derivatives* satisfy the equation.

- The question then becomes how to find the *solutions* of those equations.
Example

\[ \frac{dy}{dx} = 2x \]

One Solution: \[ y = x^2 + 1 \] (or \[ y = x^2 \])

\[ y' = 3y \]

Says: Find a function whose derivative is three times the original function.

One solution:

\[ y = 2e^{3t} \]

Check:

\[ y' = 3(2e^{3t}) = 3y \]
Verifying Solutions

Plug a proposed solution into the equation, and see if it’s true. (Remember, $y$ represents the function.)

Which are solutions to $y' = 3y$?

- $y = e^{3t}$
- $y = e^{3t} + 1$
- $y = 3t$

- Is $y = e^{3t}$ a solution to $y' = 3y$?
  
  $y' = 3e^{3t} = 3y$

  So **YES**, this is a solution to the differential equation.

- Is $y = e^{3t} + 1$ a solution to $y' = 3y$?
  
  $y' = 3e^{3t} \neq 3(e^{3t} + 1)$

  So **NO**, this is not a solution to the differential equation…

- Is $y = 3t$ a solution to $y' = 3y$?
  
  $y' = 3 \neq 3(3t)$

  So **NO**, this is not a solution to the differential equation.
Example

Confirm that \( y(t) = 1 + 2e^t \) is a solution to \( t \ y' = t \ y - t \).

Here, \( y' = 2e^t \), so \( t \ y' = t \ y - t \) becomes

\[
\begin{align*}
    t \ 2e^t & = t \left( 1 + 2e^t \right) - t \\
    2te^t & = t + 2te^t - t \\
    2te^t & = 2te^t
\end{align*}
\]

So we do have a solution!
Writing Differential Equations

Differential equations represent relationships involving rates of change.

To say the rate of change of $y$ is proportional to its current value, write

$$y' = k \, y$$

If the rate of change is twice $y$, we’d have

$$y' = 2y$$
Example

Express

“y is *decreasing* at a rate three times the value of y. (y > 0)”

\[ y' = -3y \]

“y is a function with constant rate of change 7.”

\[ y' = 7 \]

“The rate of change of y is proportional to the difference between y and 5.”

\[ y' = k(y - 5) \quad \text{Or maybe...} \quad y' = k(5 - y) \]
Example: Applications

❖ The rate of growth of a bacteria population is proportional to the size of the population:

If $y(t)$ is the population at time $t$, then $y' = k \cdot y$

❖ Bank account: Continuously compounded interest, at 1% per year.

If $A(t)$ = the amount at time $t$, then $A' = 0.01A$

❖ Chemical Reaction: • 13% per day is used up.
   • We add 7 grams per day.

If $G(t)$ is the amount in grams after $t$ days, then $G' = -0.13G + 7$
Differential Equations

• Easy to *confirm* solutions
• Harder to *find* solutions
• We will approximate solutions at a series of points, using Euler’s method.
Euler’s Method

Suppose we know: \( y'(t) \), and \( y(t_0) \).

To find \( y(t_0 + h) \):

\[
y(t_0 + h) = y(t_0) + \int_{t_0}^{t_0+h} y'(t) \, dt
\]

Let’s say \( t_1 = t_0 + h \), and now we know \( y(t_0) \) and \( y(t_1) \).

Next we find \( y(t_2) = y(t_1 + h) \), and continue, taking steps of size \( h \).
Euler’s Method

For example, with step size $h = 1$, starting at $t_0 = 0$:

\[
\begin{align*}
y(1) & \approx y(0) + (1)y'(0) \\
y(2) & \approx y(1) + (1)y'(1)
\end{align*}
\]

In general, \( y(x + h) \approx y(x) + h \, y'(x) \)

Assumes: \( y' \) is approximately constant over each subinterval.

(More accurate when \( h \) is small.)

For this class, we will usually use $h = 1$: \( y(x + 1) \approx y(x) + y'(x) \)
Example

Let \( y(0) = 1 \) and \( y' = \frac{1}{y} \). Estimate \( y \) at 1, 2, and 3.

At \( t = 0 \): \( y(0) = 1 \) and \( y'(0) = \frac{1}{y(0)} = \frac{1}{1} = 1 \)

So \( y(1) \approx y(0) + y'(0) = 1 + 1 = 2 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
</tr>
<tr>
<td>3</td>
<td>2.9</td>
</tr>
</tbody>
</table>

Note: A differential equation with an initial value (like \( y(0) = 1 \)) is called an initial value problem.
Example

Estimate $y(3)$ if $y(0) = 2$ and $y' = 3y$.

$y(1) \approx y(0) + y'(0) = 2 + 3 \cdot 2 = 8$

$y(2) \approx y(1) + y'(1) = 8 + 3 \cdot 8 = 32$

$y(3) \approx y(2) + y'(2) = 32 + 3 \cdot 32 = 128$

Actual solution: $y(t) = 2e^{3t}$

$y(1) = 2e^3 \approx 40$ \hspace{1cm} $y(2) = 2e^6 \approx 807$ \hspace{1cm} $y(3) = 2e^9 \approx 16,206$
Accuracy of Euler

• Our estimates are not great, and get worse as they go!
• Each estimate builds on the one before.
• We can improve *all* the estimates by using a smaller $h$—but we won’t do this often in 2015.
Notes:

• Euler’s method tends to be less accurate the further we go.
• Smaller step sizes increase the accuracy.
• Computers are useful for Euler’s method.
• We don’t have to start at $t = 0$!
Summary

• Defined a differential equation
• Verified solutions by plugging in functions.
• Estimated solutions using Euler’s method.
  (Adds up all the changes from the derivative.)
Group work

1. Which of the following is a solution to the differential equation $y' + y = 0$? (Show your work)

   $y = e^{2x}$          $y = xe^x$          $y = e^{-x}$          $y = 2e^x$

2. Consider the initial value problem $y' = 2y - t$, $y(1) = 3$.
   Estimate $y(2)$ and $y(3)$. 