Lab 4

1. See
   http://www.math.vt.edu/courses/math2015/labs.html

2. The quiz for lab 4 will be available this Friday.
   It will be due by next Friday (Apr 18).
This week’s hw


2. P327, #4,5 for 8, 10 a&b only
Volume

General volume (with cross section area $A(h)$)

$$Vol = \int_{a}^{b} A(h) \, dh$$

Volumes of solids with cross section as circles, $A(h) = \pi r^2(h)$

$$Vol = \int_{a}^{b} \pi r^2(h) \, dh$$
Density Functions and Probability
Density Functions and Probability

- Density functions are used to describe what proportion of a population has a certain characteristic.

- Density functions will lead us into a discussion of probability.

- Density functions used to represent information about the distribution of various quantities through the population.
What’s a Density Function?

Density functions exist only to be integrated.

**Applications:** Measures a characteristic of a population: (height of U.S. women, age of trees in a valley, scores on the last test)

If \( p(x) \) is a density function for the characteristic \( x \), then

\[
\int_a^b p(x) \, dx = \text{(fraction of the population for which } a \leq x \leq b)\]

You have to *integrate* a density function to get much meaning.

\( p(x) \) tell us how “densely” the population is distributed around certain values.

**Note:** Our examples (heights, ages, scores) are all *continuous* variables; they can take on a *range* of values. If you try to ask about a specific value, we just get ‘0’.
An Example and a Property

Age of trees in a forest is given by the density function plotted below:

Since all trees are between 0 and 75 years of age,

\[
\int_{0}^{75} p(x) \, dx = 1
\]

For a general density function,

\[
\int_{-\infty}^{\infty} p(x) \, dx = 1
\]

That is, the fraction of trees that are between 0 and 75 years old must be 1.

In our example, if \( \int_{0}^{75} p(x) \, dx = 1 \), and \( p(x) = 0 \) outside of \( 0 \leq x \leq 75 \), then

\[
\int_{-100}^{100} p(x) \, dx = 1
\]

\[
\int_{-1000}^{1000} p(x) \, dx = 1
\]

\[
\int_{-\infty}^{\infty} p(x) \, dx = 1
\]
An Example and a Property

Age of trees in a forest:

1- What fraction are 50 or younger?

\[
\int_{0}^{50} p(x) \, dx = \frac{1}{2} (50) h = 25h
\]

2- Older?

\[
\int_{50}^{75} p(x) \, dx = \frac{1}{2} (25) h = 12.5h
\]

3- What is the value of \( h \)?

\[
\int_{0}^{75} p(x) \, dx = 1
\]

\[
25h + 12.5h = 1
\]

\[
37.5h = 1 \Rightarrow h = .02\overline{6}
\]
Properties

• Already know: \( \int_{-\infty}^{\infty} p(x) \, dx = 1 \)

• \( p(x) \geq 0 \)

• Often (but not always), \( p(x) > 0 \) only on finite interval.

Example

A machine lasts up to 10 years. The figure shows the density function for the life of the machine, \( p(t) \), for the length of time it lasts.

a) What’s \( C \)?

\[
\int_{0}^{10} p(x) \, dx = 1
\]

\[
(.01)(5) + 5C = 1
\]

\[
C = \frac{1 - .05}{5} = .19
\]
Example

b) Are the machines more likely to break in 1st or 10th year?

More area is under 9–10 than under 0–1, so they are more likely to break in their 10th year.

Are they more likely to break in the first or second year?

No difference here!

The same area is under 1–2 as under 0–1, so just as many break in the first year as break in the second.
Example

What fraction lasts between 5–7 years? Between 3–6 years?

i) Lasts 5–7 years?

\[ \int_{5}^{7} p(x) \, dx = (0.19)(7 - 5) = 0.38 \]

ii) 3–6 years?

\[ \int_{3}^{6} p(x) \, dx = \int_{3}^{5} p(x) \, dx + \int_{5}^{6} p(x) \, dx = (0.01)(5 - 3) + (0.19)(6 - 5) = 0.02 + 0.19 = 0.21 \]
Probability

From the last example: We said the fraction of machines that last between 5 and 7 years was 0.38 or 38%.

We could also interpret this answer as a probability: If we randomly select a machine, there is a probability of 0.38 or 38% that the machine will fail between year 5 and year 7.

Example

What’s the probability of a machine lasting more than 5 years?

\[(5)(0.19) = 0.95\]

Less than 5 years?

\[(5)(0.01) = 0.05\]
Probability Density Function

In general, if $x$ is the variable we want to measure probability for, and $p(x)$ is the probability density function (pdf) for $x$, then the probability of the value of $x$ being between $a$ and $b$ is given by

$$\int_{a}^{b} p(x) \, dx = \begin{pmatrix} \text{probability} \\ \text{that} \\ a \leq x \leq b \end{pmatrix}$$
Example: A Spinner

A spinner (as in a game) that can land on any angle between 0 to 360°. All regions are equally likely.

What is the density function \( p(x) \) for the angle the spinner lands on?

We know it’s constant on \( 0 \leq x \leq 360 \), and 0 elsewhere:

\( h \) must be 1/360, because the integral from 0 to 360 must be 1.
Example: A Spinner

A spinner that can land on any angle between 0 to 360°. All regions are equally likely.

So we have: \[ p(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{1}{360}, & \text{for } 0 \leq x \leq 360 \\ 0, & \text{for } x > 360 \end{cases} \]

What is the probability of an angle…

1- Less than 10°?
\[ \int_0^{10} \frac{1}{360} \, dx = \frac{10}{360} = \frac{1}{36} \approx 0.0278 \]

2- Less than 90°?
\[ \int_0^{90} \frac{1}{360} \, dx = \frac{90}{360} = \frac{1}{4} = 0.25 \]

3- Less than 360°?
\[ \int_0^{360} \frac{1}{360} \, dx = \frac{360}{360} = 1 \] (That’s everything!)
Example: A Spinner

Suppose the spinner is sticky, and is much more likely to land near 90° than anywhere else. Sketch a reasonable density function for this new situation.
Summary

• Defined density functions and examined properties (like $\int_{-\infty}^{\infty} p(x) \, dx = 1$).
• Determined fraction of population which has a particular range for a variable.
• Interpreted a density function as a probability.
In Class Assignment

1. The distribution of heights, $x$, in meters, of trees is represented by the density function, $p(x)$. Calculate the fraction of trees which are:

   a) Less than 4 meters high    
   b) More than 7 meters high

   ![Diagram of density function]

   \[ p(x) = \begin{cases} 
   \frac{1}{2}, & \text{if } 1 \leq x \leq 3 \\
   0, & \text{if } x < 1 \text{ or } x > 3 
   \end{cases} \]

2. Assume $p(x)$ is the density function

   \[ p(x) = \begin{cases} 
   \frac{1}{2}, & \text{if } 1 \leq x \leq 3 \\
   0, & \text{if } x < 1 \text{ or } x > 3 
   \end{cases} \]

   Find the fraction of the population that is more than 1.5 units.

   (1) $\frac{3}{2}$  
   (2) $\frac{2}{3}$  
   (3) $\frac{3}{4}$  
   (4) $\frac{1}{2}$
The distribution of heights, $x$, in meters, of trees is represented by the density function, $p(x)$. Calculate the fraction of trees which are:

a) Less than 4 meters high  

$$\int_{0}^{4} p(x) \, dx = 4 \times 0.05 = 0.20 \text{ or } 20\%$$

b) More than 7 meters high  

$$\int_{7}^{20} p(x) \, dx = 13 \times 0.05 = 0.65 \text{ or } 65\%$$
In Class Assignment

Assume $p(x)$ is the density function

$$p(x) = \begin{cases} 
\frac{1}{2}, & \text{if } 1 \leq x \leq 3 \\
0, & \text{if } x < 1 \text{ or } x > 3 
\end{cases}$$

Find the fraction of the population that is more than 1.5 units.

(1) $\frac{3}{2}$  (2) $\frac{2}{3}$  (3) $\frac{3}{4}$  (4) $\frac{1}{2}$

$$\int_{1.5}^{3} p(x)\,dx = \int_{1.5}^{3} \frac{1}{2}\,dx = \frac{3}{4} \text{ or } 75\%$$
Lesson 20

Probability and Cumulative Distribution Functions
Recall

If $p(x)$ is a density function for some characteristic of a population, then

$$\int_{a}^{b} p(x) \, dx = \left( \begin{array}{c} \text{fraction of the} \\ \text{population for} \\ \text{which } a \leq x \leq b \end{array} \right)$$

We also know that for any density function,

$$\int_{-\infty}^{\infty} p(x) \, dx = 1$$
Recall

We also interpret density functions as probabilities:

If $p(x)$ is a probability density function (pdf), then

$$
\int_{a}^{b} p(x) \, dx = \begin{cases} 
\text{probability} \\
\text{that} \\
a \leq x \leq b
\end{cases}
$$
Suppose we have a density function for final grades of 2015 students:

Most common 2015 grade?

Looks like a C.
Example

Suppose we have a density function for final grades of 2015 students:

Rough estimate a randomly selected student earns an A:
Example

Suppose we have a density function for final grades of 2015 students:

Rough estimate a randomly selected student earns an A:

90–100 looks like a little less than 1/10 of total area, so a little under 10% probability.
Cumulative Distribution Function

Suppose $p(x)$ is a density function for a quantity.

The *cumulative distribution function* (cdf) for the quantity is defined as

$$P(x) = \int_{-\infty}^{x} p(t) \, dt$$

Gives:

- The proportion of population with value less than $x$
- The probability of having a value less than $x$. 
Example: A Spinner

Last class: A spinner that could take on any value
\(0^\circ \leq x \leq 360^\circ\).

Density function: \(p(x) = 1/360\) if \(0 \leq x \leq 360\), and 0 everywhere else.
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = 1/360$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = 1/360$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 
\end{cases}$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = 1/360$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF: We integrate 0 until we reach $x = 0$.

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 0, & \text{if } x < 0 \end{cases}$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = 1/360$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 0, & \text{if } x < 0 \end{cases}$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = \frac{1}{360}$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 0, & \text{if } x < 0 \\ \int_{0}^{x} \frac{1}{360} \, dt, & \text{otherwise} \end{cases}$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = \frac{1}{360}$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 0, & \text{if } x < 0 \\ \int_{0}^{x} \frac{1}{360} \, dt = \frac{x}{360} & \text{if } x \geq 0 \end{cases}$$
Example: A Spinner

Last class: A spinner that could take on any value
$0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = \frac{1}{360}$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{x}{360}, & \text{if } 0 \leq x \leq 360 
\end{cases}$$

$$\int_{0}^{x} \frac{1}{360} \, dt = \frac{x}{360}$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = \frac{1}{360}$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{360}, & \text{if } 0 \leq x \leq 360 \end{cases}$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = \frac{1}{360}$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{360}, & \text{if } 0 \leq x \leq 360 \end{cases}$$

Probability $x < 360$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = 1/360$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{x}{360}, & \text{if } 0 \leq x \leq 360 \\
1, & \text{if } x > 360
\end{cases}$$
Example: A Spinner

Last class: A spinner that could take on any value $0^\circ \leq x \leq 360^\circ$.

Density function: $p(x) = 1/360$ if $0 \leq x \leq 360$, and 0 everywhere else.

CDF:

$$P(x) = \int_{-\infty}^{x} p(t) \, dt = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{360}, & \text{if } 0 \leq x \leq 360 \\ 1, & \text{if } x > 360 \end{cases}$$
Example: A Spinner

Density Function:

Cumulative Distribution Function: