Lesson 17

Volumes of Solids of Revolution
After measuring a weird object 3 feet tall, we find that the cross sectional areas are given as follows:

<table>
<thead>
<tr>
<th>h (feet)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (ft²)</td>
<td>2</td>
<td>3</td>
<td>1.5</td>
<td>1</td>
</tr>
</tbody>
</table>

Estimate the volume of the object. **Show what you are doing.**

\[
\sum \text{bottom areas } \times dh = (2 \times 1) + (3 \times 1) + (1.5 \times 1) = 6.5
\]

\[
\sum \text{top areas } \times dh = 3 + 1.5 + 1 = 5.5
\]

\[
\text{Average} = \frac{6.5 + 5.5}{2} = 6 \text{ ft}^3
\]
Blobs in Space

Volume of a blob:

Cross sectional area at height $h$: $A(h)$

$$\text{Volume} = \int_a^b A(h) \, dh$$

Example

Solid with cross sectional area $A(h) = 2h$ at height $h$. Stretches from $h = 2$ to $h = 4$. Find the volume.

$$\int_2^4 2h \, dh = h^2 \bigg|_2^4 = 16 - 4 = 12 \text{ cubic units}$$
Solids of Revolution

Rotate a region about an axis.

Such as: Region between $y = x^2$ and the $y$-axis, for $0 \leq x \leq 2$, about the $y$-axis.

For solids of revolution, cross sections are circles, so we can use the formula

$$\text{Volume} = \int_a^b \pi (r(h))^2 \, dh$$

just like for the tree trunks.

Usually, the only difficult part is determining $r(h)$. A good sketch is a big help.
Solids of Revolution

Rotate a region about an axis.

Such as: Region between $y = \sqrt{x}$

and the $x$-axis, for $1 \leq x \leq 4$,
about the $x$-axis.

How to form a torus as a solid of rotation?

The circle is revolved around the $x$-axis.

The resulting solid of revolution is a torus.
Example

Rotate region between $x = \sin(z), \ 0 \leq z \leq \pi,$ and the $z$ axis about the $z$ axis.

We need to figure out what the resulting solid looks like.

Aside: Sketching Revolutions

1. Sketch the curve; determine the region.
2. Sketch the reflection over the axis.
3. Sketch in a few “revolution” lines.
Example

Rotate region between \( x = \sin(z) \), \( 0 \leq z \leq \pi \), and the \( z \)-axis about the \( z \)-axis.

Now find \( r(z) \), the radius at height \( z \).

(Each slice is a circle.

So \( r(z) = \sin(z) \).

What are the limits? The variable is \( z \), so the limits are in terms of \( z \)…

Upper limit: \( z = \pi \)

Lower limit: \( z = 0 \)

Volume \( = \int_{0}^{\pi} \pi \left( r(z) \right)^2 dz = \int_{0}^{\pi} \pi \left( \sin(z) \right)^2 dz = 4.9 \)

We must do this numerically!
Example

Revolve the region under the curve $y = 3e^{-x}$, for $0 \leq x \leq 1$, about the $x$ axis.

(Since we rotate about $x$, the slices are perpendicular to $x$. So $x$ is our variable.)

First, get the region:

Next, revolve the region:

When we rotate, this will become a radius.

Radius at point $x$ is $3e^{-x}$.

Limits are from $x = 0$ to $x = 1$.

Volume:

$$
\int_a^b \pi (r(x))^2 \, dx = \int_0^1 \pi \left(3e^{-x}\right)^2 \, dx = \pi \int_0^1 9e^{-2x} \, dx = \pi \left[ -\frac{9}{2} e^{-2x} \right]_0^1 \approx 12.22 \text{ units}^3.
$$
Remember

- Slices perpendicular to axis of rotation.
- Radius is then a function of position on that axis.
- Therefore rotating about $x$ axis gives an integral in $x$; rotating about $z$ gives an integral in $z$. 
Example

Rotate $z = x^2$, from $x = 0$ to $x = 4$, about first the $x$-axis, and then $z$.

First, sketch:

Next, rotate about the $x$ axis:

Next, find $r(x)$: $r(x) = x^2$  

Limits: $x = 0$ to $x = 4$

Volume:

$$
\int_0^4 \pi \left( x^2 \right)^2 dx = \pi \int_0^4 x^4 dx = \pi \frac{x^5}{5} \bigg|_0^4 = \frac{\pi}{5} \left( 4^5 - 0 \right) \approx 643.4 \text{units}^3
$$
Example

Now rotate $z = x^2$, from $x = 0$ to $x = 4$, about $z$ axis.

Radius: $r(z) = \sqrt{z}$.

Limits: $z = 0$ to $z = 16$.

Volume:

$$\int_0^{16} \pi (\sqrt{z})^2 \, dz = \pi \int_0^{16} z \, dz = \pi \left[ \frac{z^2}{2} \right]_0^{16} = \frac{\pi}{2} (16^2 - 0) \approx 402.12 \text{ units}^3$$
Summary

- Described how a plane region rotated about an axis describes a solid.
- Found volumes of solids of revolution using
  \[ \int_{a}^{b} \pi (r(h))^2 \, dh = \int_{a}^{b} \pi r(h)^2 \, dh \]
- Determined the variable as given by the axis of rotation.
In Class Assignment

Find the volume resulting when the curve $y = 3x$, on $1 \leq x \leq 3$, is rotated...

...about the $x$-axis.

...about the $y$-axis.

Include pictures.