About Test 2

1. Test 2
   Time: Thursday, March 27, 2008 at 7 PM.
   Location: Der 1090


3. We will review for the test next Monday.
   Next Wed’s class would be in Question-Answer form.
1. A constant function and e^x are walking on Broadway. Then suddenly the constant function sees a differential operator approaching and runs away. So e^x follows him and asks why the hurry..

"Well, you see, there's this differential operator coming this way, and when we meet, he'll differentiate me and nothing will be left of me...!"

"Ah," says e^x, "he won't bother ME, I'm e to the x!" and he walks on. Of course he meets the differential operator after a short distance

e^x: "Hi, I'm e^x"
diff.op.: "Hi, I'm d/dy"

2. Person 1: What's the indefinite integral of 1/cabin?
   Person 2: Log cabin.
   Person 1: No, a houseboat – you forgot to add the C (sea)!
1. Correct notations are important

2. Indefinite integral is a family of functions (so don’t forget the constant C)
   Definite integral is a number
Improper Integral Review

• We recognize as *improper* integrals with one or both limits infinite.
• We evaluate these by taking the limit as the upper/lower limit increases/decreases.
• Improper integrals may *diverge* (fail to exist).
• Integrals with *both* limits infinite are a little tricky.
Example

\[
\int_{1}^{\infty} \! e^{-2x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \! e^{-2x} \, dx = \lim_{b \to \infty} \! \left. \frac{e^{-2x}}{-2} \right|_{1}^{b} = -\frac{1}{2} \left( \frac{1}{e^{2b}} - \frac{1}{e^{2(1)}} \right)
\]

\[
= -\frac{1}{2} \left( \left( \lim_{b \to \infty} \! \frac{1}{e^{2b}} \right) - \frac{1}{e^{2(1)}} \right) = -\frac{1}{2} \left( 0 - \frac{1}{e^{2}} \right) = \frac{1}{2e^{2}}
\]

\[
\int_{1}^{\infty} \frac{1}{t} \, dt = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{t} \, dt = \lim_{b \to \infty} \left( \ln |t| \right|_{1}^{b}
\]

\[
= \lim_{b \to \infty} (\ln b - \ln 1) = \lim_{b \to \infty} \ln b = \infty (\text{diverges}!)
\]
Lesson 15

Analyzing Antiderivatives
Finding $F$ from $F'$

Recall:  

$$F(b) - F(a) = \int_a^b F'(x) \, dx$$  

Or:  

$$F(b) = F(a) + \int_a^b F'(x) \, dx$$

Example  Suppose $F'(x) = 2x + 1$ and $F(0) = 1$. Find $F(2)$.

$$F(2) = F(0) + \int_0^2 F'(x) \, dx = 1 + \int_0^2 2x + 1 \, dx = 1 + (x^2 + x) \bigg|_0^2 = 1 + (4 + 2) = 7$$

Practice Problems

With your group, work on the two practice problems on pp. 1–2 in your notes.

Both are about finding antiderivatives, or recovering $f$ from $f'$. 
Practice 1

Given the graph of $F'$ as shown on the right, and the fact that $F(0) = 2$, find the following: $F(0), F(1), F(2), F(3), F(4), F(5)$

$F(0) = 2$
$F(1) = 2 + 1 = 3$
$F(2) = 3 + 0.5 = 3.5$
$F(3) = 3.5 - 0.5 = 3$
$F(4) = 3 - 0.5 = 2.5$
$F(5) = 2.5 + 0.5 = 3$

Sketch the graph of $F$ on the interval $[0, 5]$, based on its derivative above and the points you have found.

$F'$ decreasing means $F$ is concave down

$F$ is concave down
The graph of \( g'(x) \) is shown below:

- \( g \) is increasing on \([0, 2.5], [3.6, 4]\)
- \( g \) is decreasing on \([2.5, 3.6]\)

\( g \) increases where the derivative is + and decreases where it is -

- Local max: at \( x = 2.5 \)
- Local min: at \( x = 3.6 \)

Largest value on \([0, 4]\)? at \( x = 2.5 \)
Smallest value on \([0, 4]\)? at \( x = 0 \)
Substitution and Differential Equations

We want to find all functions $y(t)$ such that $y'(t) = ky(t)$

Start by dividing by $y(t)$ on both sides to get $\frac{y'(t)}{y(t)} = k$

Now find the antiderivative on both sides with respect to $t$:

$$\int \frac{y'(t)}{y(t)} dt = \int k \, dt$$

The right side is easy; we just get $k \, t + C_1$.

On the left, we make the substitution $u = y(t)$, so that $du = y'(t) \, dt$

and the integral becomes $\int \frac{du}{u} = \ln|u| + C_2 = \ln|y(t)| + C_2$

Combining the constants with $C = C_1 - C_2$, we get $\ln|y| = k \, t + C$.

This can be solved as $y(t) = \pm e^{kt+C} = \pm e^C e^{kt}$

If we let $A = \pm e^C$, we have our familiar solution, $y(t) = Ae^{kt}$.
Antiderivative Practice

Problem 1 \[ \int (q^2 + 5)^3 \, dq \] Use algebra:

\[ \int (q^2 + 5)^3 \, dq = \int q^6 + 15q^4 + 75q^2 + 125 \, dq = \frac{1}{7} q^7 + 3q^5 + 25q^2 + 125q + C \]

Problem 2 \[ \int q(q^2 + 5)^3 \, dq \] Make the substitution \( u = q^2 + 5 \), \( du = 2q \, dq \):

\[ \int q(q^2 + 5)^3 \, dq = \frac{1}{2} \int (q^2 + 5)^3 \, 2q \, dq = \frac{1}{2} \int u^3 \, du = \frac{1}{8} u^4 + C = \frac{1}{8} (q^2 + 5)^4 + C \]

Problem 3 \[ \int e^{3-y} \, dy \] use the substitution \( u = 3 - y \), so that \( du = -dy \):

\[ \int e^{3-y} \, dy = -\int e^u \, du = -e^u + C = -e^{3-y} + C \]

Problem 4 \[ \int \frac{x + 1}{3\sqrt{x}} \, dx \] Simplify algebraically first:

\[ \int \frac{x + 1}{3\sqrt{x}} \, dx = \int \frac{x}{x^{1/3}} + \frac{1}{x^{1/3}} \, dx = \int x^{2/3} + x^{-1/3} \, dx = \frac{3}{5} x^{5/3} + \frac{3}{2} x^{2/3} + C \]
Antiderivative Practice

Problem 5 \( \int \cos(\theta)\sqrt{1+\sin(\theta)} \, d\theta \) Make the substitution \( u = 1 + \sin(\theta) \)
so \( du = \cos(\theta) \, d\theta \) to get the following:

\[ \int \cos(\theta)\sqrt{1+\sin(\theta)} \, d\theta = \int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1+\sin(\theta))^{3/2} + C \]

Problem 6 \( \int \frac{3}{2t} \, dt \) Simplify and evaluate:

\[ \int \frac{3}{2t} \, dt = \frac{3}{2} \int \frac{1}{t} \, dt = \frac{3}{2} \ln|t| + C \]

Problem 7 \( \int_2^9 3\sqrt{s} \, ds \) Use the fundamental theorem:

\[ \int_2^9 3\sqrt{s} \, ds = \int_2^9 3s^{1/2} \, ds = 2s^{3/2}\bigg|_2^9 = 2(9)^{3/2} - 2(2)^{3/2} = 54 - 4\sqrt{2} \]