Welcome to

**Elementary Calculus with Trig II**

**CRN(13828)**

Instructor: Quanlei Fang
Department of Mathematics, Virginia Tech, Spring 2008
Be sure to read the course contract

- Contact Information
- Text
- Grading
- In-class work/ Hw/ Lab work/ Tests
- Make-up policy
- Honor system
- Special needs
Read the syllabus ....

- Do homework regularly
- Prepare for the quizzes
- Mark the test dates in your calendar
- Do the Lab assignments and take lab quizzes in time
Questions?
Lets get it started!
Riemann Sums and Accumulated Change
Math 1016

Given data, find the rate of change.

Math 2015

Given a rate of change, find the total change.

Example: Velocity and Distance Traveled

Goal: If we have information about the rate at which a quantity is changing over time, we wish to be able to determine the total amount it will have changed after a given amount of time.
Recovering Distance From Velocity

If we travel

2 feet per second for 4 seconds,

Then we traveled a total of 8 feet.

Why?
Example

• Suppose that you travel at a speed of:
   60 mph for 2 hours, then
   40 mph for 30 minutes, then
   50 mph for 3 hours.

• What is the total distance you have traveled?
  \[60 \times 2 + 40 \times 0.5 + 50 \times 3 = 290\]

• When the speed stays exactly the same for a period of time, the problem involves no calculus, only simple multiplication and addition.

• So, what if the speed is changing?
Recovering Distance From Velocity

Model Train Data:

<table>
<thead>
<tr>
<th>$t$ (sec)</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$ (ft/sec)</td>
<td>2</td>
<td>3</td>
<td>3.5</td>
</tr>
</tbody>
</table>

How far does it travel between $t = 4$ and $t = 8$?
Recovering Distance From Velocity

Three ways to answer:

- Use the first velocity: \((2 \text{ feet/sec})(4 \text{ sec}) = 8 \text{ feet}\)
- Use the middle velocity: \((3 \text{ feet/sec})(4 \text{ sec}) = 12 \text{ feet}\)
- Use the final velocity: \((3.5 \text{ feet/sec})(4 \text{ sec}) = 14 \text{ feet}\)
A Useful Aside:

Suppose we know the train is speeding up.

Romola Claims:

It *must* have gone *at least* 8 feet,
and *less than* 14 feet.

Is she right? Why or why not?

- Using first velocity: 8 feet
- Using middle velocity: 12 feet
- Using final velocity: 14 feet
She’s Right!

*If* it’s speeding up, the first speed is the *lowest*, and the last is the *highest*.

If it’s always faster than 2 ft/sec, the train travels more than

\[(2 \text{ feet/sec})(4 \text{ sec}) = 8 \text{ feet}\]

If it’s always going slower than 3.5 ft/sec, the train travels less than

\[(3.5 \text{ feet/sec})(4 \text{ sec}) = 14 \text{ feet}\]
But what if it’s not speeding up?

Then anything might happen!
The total distance might be less than 8 feet…
Or more than 14 feet!
Subintervals

<table>
<thead>
<tr>
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<th>6</th>
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<td>2</td>
<td>3</td>
<td>3.5</td>
</tr>
</tbody>
</table>

How far just from $t = 4$ to $t = 6$?
- Using the starting $v$:
- Using the ending $v$:

What about from $t = 6$ to $t = 8$?
- Using the starting $v$:
- Using the ending $v$:

The total distance traveled is approximately...

If we use the first velocity in each subinterval: $(2 \times 2) + (3 \times 2) = 10$ ft. Called a **Left-hand sum**.

If we use the last velocity in each subinterval: $(3 \times 2) + (3.5 \times 2) = 13$ ft. Called a **Right-hand sum**.
Left and Right hand sums are both examples of Riemann sums.

Georg Friedrich Bernhard Riemann
September 17, 1826–July 20, 1866
Questions:

1) If the train is speeding up, is either sum an over or underestimate of the distance?
   The left-hand sum is an underestimate;
   The right-hand sum is an overestimate.

2) What if the train was slowing down instead?
   The left-hand sum would be an overestimate;
   The right-hand sum would be an underestimate.

3) Are these new estimates better or worse than our first estimate (using one interval)?
   Probably better! (They use more data, and the velocity does not change much over small intervals.)
Improving our Estimates

We can improve our estimates by:

• averaging left and right hand sums, or
• using more subintervals.
More Data

Suppose we collect additional data about our train’s velocity:

<table>
<thead>
<tr>
<th>$t$</th>
<th>4</th>
<th>5.5</th>
<th>6</th>
<th>6.2</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$</td>
<td>2</td>
<td>2.7</td>
<td>3</td>
<td>3.1</td>
<td>3.3</td>
<td>3.5</td>
</tr>
</tbody>
</table>
# Left-Hand Sum Table

<table>
<thead>
<tr>
<th>Interval</th>
<th>$t$</th>
<th>$\Delta t$</th>
<th>$v$</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>[4, 5.5]</td>
<td>1.5</td>
<td>2</td>
<td>3</td>
</tr>
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Which subinterval
Subinterval width
Traveled in this subinterval

What $t$ values
Velocity for this subinterval
## Left-Hand Sum

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<td>3</td>
</tr>
<tr>
<td>#2</td>
<td>[5.5, 6]</td>
<td>0.5</td>
<td>?</td>
<td>1.35</td>
</tr>
<tr>
<td>#3</td>
<td>[6, 6.2]</td>
<td>0.2</td>
<td>3</td>
<td>0.6</td>
</tr>
<tr>
<td>#4</td>
<td>[6.2, 7]</td>
<td>?</td>
<td>3.1</td>
<td>2.48</td>
</tr>
<tr>
<td>#5</td>
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<td>1</td>
<td>3.3</td>
<td>3.3</td>
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Total: 10.73
## Right-Hand Sum

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<td>4.05</td>
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<td>0.5</td>
<td>3</td>
<td>1.5</td>
</tr>
<tr>
<td>#3</td>
<td>[6, 6.2]</td>
<td>0.2</td>
<td>3.1</td>
<td>0.62</td>
</tr>
<tr>
<td>#4</td>
<td>[6.2, 7]</td>
<td>0.8</td>
<td>3.3</td>
<td>2.64</td>
</tr>
<tr>
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</table>

Total: 12.31

Average of LHS and RHS: 11.52 feet
Pictures

On the velocity graph:

Velocity is height

$\Delta t$ is width

So $v(t) \Delta t$ is the area of a rectangle!

Note: Height depends on which velocity we use!
Pictures
Riemann sum rectangles, $\Delta t = 4$ and $n = 1$:

Riemann sum rectangles, $\Delta t = 2$ and $n = 2$:
Pictures

Riemann sum rectangles, $n = 5$ ($\Delta t$ varies)
Accumulated Change

• Velocity is the rate of change of position.
• We can apply Riemann sums to any other rate of change.
• A Riemann sum applied to a rate of change estimates the total change over the interval.
Example: A Reservoir

Filling at variable rate, starting with 1,000 gallons at noon.

Total volume at 1:00?

<table>
<thead>
<tr>
<th>Time (minutes past noon)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate (gal/min)</td>
<td>5</td>
<td>16</td>
<td>32</td>
<td>30</td>
</tr>
</tbody>
</table>
Example: A Reservoir

First, estimate the change:

Left-hand sum: \(20(5+16+32) = 20 \times 53 = 1060\)

Right-hand sum: \(20(16+32+30) = 20 \times 78 = 1560\)

Best Estimate: **Average: \((1060+1560)/2 = 1310\)**

<table>
<thead>
<tr>
<th>Time (minutes past noon)</th>
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Total volume at 1:00 is the change *added to the original volume*:

\[1,000 + 1,310 = 2,310 \text{ gallons}\]
Summary

• Approximate total change from rate of change using left/right hand sums.
• For *increasing functions*, left hand sums underestimate and right hand overestimate.
• Left hand sum: multiply function at left of each subinterval by width of subinterval.
• Riemann sums look like areas of rectangles.
Review

• The main question answered in §5.1 is:

*If we know the _____rate of_____ change, how can we estimate the __________ change?*
Review

- If the speed is changing continuously, we can approximate the total distance traveled by assuming that the velocity is **constant** over small intervals of time.
Review

• If we use a smaller time interval, the upper and lower estimates get ________ (a)

(a) closer together
(b) farther apart
Assignment

Go to the Math 2015 Course Homepage:
http://www.math.vt.edu/courses/math2015

1. Read the Announcements - after every class
2. Read and complete the Assignment before next Monday’s class.
3. There will be no in-class assignment today.