Riemann Sums and Accumulated Change

We will study how to measure the total change in a quantity from examining its rate of change, beginning with a major example: If you happen to know the velocity of an object at different times, can you estimate its position?

Recovering Distance from Velocity

If an object travels at 2 ft/sec for a total of 4 seconds, how far does it travel? Since distance = (rate) (time), it must have traveled __ ____.

Suppose instead we had the following data about the velocity of a model train at the given times:

<table>
<thead>
<tr>
<th>time ( t ) (seconds)</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity ( v ) (ft/second)</td>
<td>2</td>
<td>3</td>
<td>3.5</td>
</tr>
</tbody>
</table>

How can we estimate the distance traveled between \( t = 4 \) and \( t = 8 \)? Here are three estimates:

- Use the beginning velocity: Then we have \((2 \text{ ft/sec})(4 \text{ seconds}) = 8 \text{ feet.}\)
- Use the ending velocity: Then we have \((3.5 \text{ ft/sec})(4 \text{ seconds}) = 14 \text{ feet.}\)
- Use the middle velocity: Then we have \((3 \text{ ft/sec})(4 \text{ seconds}) = 12 \text{ feet.}\)

A useful aside: Suppose we know that the train is speeding up between \( t = 4 \) and \( t = 8 \) seconds. Romola looks at this data, and says:

  It must have gone at least 8 feet, and less than 14 feet.

Critique Romola’s response. Is she right or wrong, and why?

What if we didn’t know that the train was always speeding up? Would that change your answer?

Using Subintervals

Here’s another approach to estimating the distance traveled:

Let’s ask how far the train moved just between \( t = 4 \) and \( t = 6 \), a subinterval of our original time interval. We get:
• Using the starting \( v \): (2 ft/sec)(2 sec) = 4 ft
• Using the ending \( v \): (3 ft/sec)(2 sec) = 6 ft

Similarly we can estimate the distance traveled between \( t = 6 \) and \( t = 8 \):
• Using the starting \( v \): (3 ft/sec)(2 sec) = 6 ft
• Using the ending \( v \): (3.5 ft/sec)(2 sec) = 7 ft

Finally, we can add our results from the two intervals: total distance traveled is about
• 4 + 6 = 10 feet, if we use starting \( v \) in each subinterval, or
• 6 + 7 = 13 feet, if we use the ending \( v \).

The estimate using \( v \) on the left side of each subinterval is called a left-hand sum, and the estimate using \( v \) on the right side of each subinterval is called right-hand sum.

Both of these are examples of a Riemann sum, in which we multiply a function value times the width of each interval and add up all the results.

**Question:** If the train is still only speeding up, which of these is an overestimate and which an underestimate of the distance traveled?

What if the data represented a train slowing down instead of speeding up?

**Question:** Do you think these new estimates are probably more or less accurate than our first estimates? Why?

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**Improving our Estimates**

Here are two possible ways to get a better estimate:
• Average our two estimates (left and right hand sums).
• Use more subintervals!

Of course using more subintervals requires more data, so let’s suppose we gathered some more velocity readings:

\[
\begin{array}{c|ccccccc}
 t \text{ (sec)} & 4 & 5.5 & 6 & 6.2 & 7 & 8 \\
v \text{ (ft/sec)} & 2 & 2.7 & 3 & 3.1 & 3.3 & 3.5 \\
\end{array}
\]

To help us keep track of the sum, it’s convenient to use a table. For each subinterval, we need to multiply the width of the subinterval (time) by a velocity at the beginning (left hand sum) or end of the subintervals (right hand sum).
Let’s start with a left-hand sum:

So for example, subinterval #1 consists of \( t \) values in \([4, 5.5]\), so its width is \( 5.5 - 4 = 1.5 \). The velocity at the left hand side of the subinterval is 2 ft/sec, so our first line is

<table>
<thead>
<tr>
<th>Interval</th>
<th>( t )</th>
<th>( \Delta t )</th>
<th>( v )</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>([4, 5.5])</td>
<td>1.5</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

(Here we use the Greek letter \( \Delta \) to represent “change in” \( t \); in other words, the width of the time interval.) Here’s the rest of the table, with a few missing entries to fill in:

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<td>1.5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>#2</td>
<td>([5.5, 6])</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#3</td>
<td>([6, 6.2])</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#5</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Total: ____________

And here’s the right hand sum:

<table>
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<tr>
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</tr>
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</tr>
<tr>
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Total: ____________

We can of course take the average of the left and right hand sums:

**Pictures**

We can also look at Riemann sums graphically. We first used a left hand, right hand, and a midpoint sum with just one subinterval for our velocity function on \([4, 8]\). The velocity function \( v(t) \) at right gives velocity as \( y \) values, or heights on the graph, while time intervals appear as ranges of \( x \) values, or widths. Each term in the sum has a height (given by the velocity function at some point) and a width (the time interval), so
each term represents a rectangle (height times width).

Here are what the three rectangles look like, plotted with $v(t)$:

![Left Hand](image1.png) ![Midpoint](image2.png) ![Right Hand](image3.png)

We also did left and right hand sums using the subintervals $[4, 6]$ and $[6, 8]$. Each subinterval has its own velocity times time, so each subinterval gets its own rectangle:

![Left Hand](image4.png) ![Right Hand](image5.png)

Note that in the case of the left hand sum, since we get the height from the left side of each subinterval, each rectangle touches the velocity curve on the left. For the right hand sum, each rectangle touches the curve on the right side.

Finally, when we used five subintervals, we get five rectangles:

![Left Hand](image6.png) ![Right Hand](image7.png)

**Accumulated Change**

Velocity is really the rate at which the position of an object is changing. (Note that it is measured in units like feet/second or miles/hour!) The accumulated change is the total change in position, or how far the object moved.

In general, any time we calculate a Riemann sum for a rate of change, we approximate the total change.

**Example:** A reservoir is being filled at a variable rate, in gallons per minute. We start at noon, and the rate at which water is being put into the reservoir is shown in gallons per minute at times after 12:00 in the following table:
If there is initially 1,000 gallons in the tank at noon, approximately how much is in the tank at 1:00?

First, calculate the total change in the volume from the rate of change:

Left Hand Sum:

Right Hand Sum:

Average: _________ gallons added to the reservoir.

So at 1:00 (after one hour), we have added about _____ gallons, and so now have ____________________ gallons.

**Summary**

We have found that:

- Given a rate of change (for example, velocity), we can approximate the total change (for example, distance traveled) by using a left or right hand sum, which are examples of Riemann sums.

- If we know for certain that the function is always increasing, then the approximation to distance traveled is an overestimate if we use a right hand sum, and an underestimate if we use a left hand sum. (The reverse is true for a decreasing function.)

- To create a left hand sum, we subdivide up an interval into subintervals. We multiply the width of each subinterval by the value of the function (the rate of change) on the left side of the subinterval. A right hand sum is the same, but using the right side of each subinterval.

- Multiplying height (from the function) times width on the graph is the same as finding the area of a little rectangle. Left hand sums have rectangles that touch the graph on their left side, and right hand sums have rectangles that touch the graph on the right.

The most important point for today is the following:

If we have the rate of change of some quantity (position, population, whatever), then a left or right hand sum will approximate the total change in the quantity over a time interval.

**Homework:** You need to visit your class website. You will find important information here, including assignments for each day.