I. For each of the following relations, determine:
   a. if the relations are functions.
   b. if they are functions, then find the range.
   c. if the functions are 1-1.
   d. if the functions are onto.
   e. if the functions are a 1-1 correspondence. If it is a 1-1 correspondence, find its inverse function.

1. Let \( X = \{ \Box, \Box, \oplus \} \), \( Y = \{1, 2, 3\} \)
   \( f : X \rightarrow Y \) is given by \( f(\Box) = 2, f(\Box) = 3, f(\oplus) = 1 \)

2. Let \( f:Z \rightarrow Z \) is given by \( f(x,y) = x + y \).

3. Let \( U = \{1,2\} \). \( f : P(U) \rightarrow \{ S | S \text{ is a partition of an element in } P(U) \} \) is given by \( f(A) = \text{a partition of } A \) (for every \( A \subseteq U \)). Hint: Write out all the subsets of U and all the partitions for each subset.

4. Let \( \{0,1\}^3 \) be the set of strings of 0s and 1s of length 3. Define \( f: \{0,1\}^3 \rightarrow \{0,1\}^3 \), \( f(s) = \text{the string obtained by writing the characters of } s \text{ in reverse order.} \)

5. Let \( f: [2,3] \rightarrow [2,5] \) be defined by the graph below.

II. Let \( A = \{2, 5, 6, 7\} \) and \( B = \{1, 2, 3, 4, 5\} \). Let \( f : A \rightarrow B \) and \( g : B \rightarrow A \) where \( f = \{(2,2), (5,4), (6,5), (7,1)\} \) and \( g = \{(1,2),(2,7),(3,5),(4,6),(5,5)\} \)

Find: 1. \( f \circ g \). Write as a set of ordered pairs.

2. Does \( g^{-1} \) exist? Why or why not?

3. Does \( (f \circ g)^{-1} \) exist? Why or why not?
III.1. Let \( f, g : \mathbb{R} \to \mathbb{R} \) where \( g(x) = 1 - x + x^2 \) and \( f(x) = ax + b \). If \( (g \circ f)(x) = 9x^2 - 9x + 3 \), find \( a \) and \( b \).

2. Define \( f \) and \( g \) on the family tree given below by \( f(x) = \) the spouse of \( x \) and \( g(x) = \) the mother of \( x \).

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George & Jane                  Frank & Eleanor
|                             |                             |
Richard & Lucille             Frederick & Ethel
|                             |                             |
Luci & Tony                   |
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Find:
1. \( (f \circ f)(\text{Luci}) \)
2. \( (g \circ f)(\text{Luci}) \)
3. \( (f \circ g)(\text{Luci}) \)
4. \( (g \circ g)(\text{Luci}) \)

IV. 1. A rural mail route has exactly 143 boxes. How many letters must be delivered to boxes on this route to guarantee that some box will receive at least 3 letters?

2. Let \( S \subseteq \mathbb{Z}^+ \), where \( S = \{ s \mid s \equiv 37 \} \). Must \( S \) contain two elements that have the same remainder upon division by 36?

3. A penny collection contains twelve 1967 pennies, seven 1968 pennies, and eleven 1971 pennies. If you are to pick some pennies without looking at the dates, how many must you pick to be sure of getting at least six pennies from the same year?

V. 1. If \( A = \{1, 2, 3, 4, 5\} \), what is the cardinality of \( P(A) \) (power set of \( A \))?

2. Let \( A = \{ 1, 4, 9, 16, \ldots \} = \{ n^2 \mid n \in \mathbb{Z}^+ \} \). Show that \( A \) is countable.

3. Prove that if \([a,b]\) and \([c,d]\) are any two intervals of real numbers with the same length (that is, \( b - a = d - c \)), then these intervals have the same cardinality.
Answers:

I. 1. function, range = \{1, 2, 3\}, 1 – 1, onto, 1 – 1 correspondence
2. function, range = \mathbb{Z}, not 1 – 1, onto, not 1 – 1 correspondence
3. not a function
4. function, range
5. function, range = \{2, 0\} \cup \{0, 5\}, 1 – 1, not onto, not 1 – 1 correspondence
   (If you consider that all the points are shown on the graph, then this is not a function. \(f(0.1)\) is not defined.

II. 1. \{(1, 2), (2, 1), (3, 4), (4, 5), (5, 4)\}
2. no \(g\) is not 1 – 1
3. no \(f \circ g\) is not onto

III. 1. \(a = 3, b = -1\)

IV. 1. 287 2. yes \(s = 1 \text{ and } s = 37\) 3. 16

V. 1. 32
2. Let \(f : \mathbb{Z}^+ \rightarrow A, f(n) = n^2\)
   \(n_1^2 = n_2^2\)
   \(n_1 = n_2\) so \(f\) is 1-1.

   Let \(a = n^2\), then \(\sqrt{a} = n\). Since \(a \in A, \exists k \in \mathbb{Z}, a = k^2\)
   \(n = \sqrt{k^2} = k \in \mathbb{Z}\) so \(f\) is onto.

3. Let \(f : [a,b] \rightarrow [c,d], f(x) = (x \cdot a) + c\)
   \(f(a) = c, f(b) = d = (b \cdot a) + c\)
   \(y = (x \cdot a) + c\)
   \(x_1 \cdot a + c = x_2 \cdot a + c\)
   \(x_1 = x_2\) so \(f\) is 1-1.

   \(y = (x \cdot a) + c\)
   \(x = y + a \cdot c\). Since \(y \in [c,d], c \leq y \leq d\)
   \(c + a \cdot c \leq y + a \cdot c \leq d + a \cdot c\)
   \(a \cdot x \leq b\) \(b = d + a \cdot c\)
\[ x \geq y \text{ is onto.} \]