Chapter 10

I. 1. Let \( A = \{2, 4, 6\} \). Define a binary relation from \( A \) to \( A \) as follows:
\[ R = \{(2, 6), (4, 2), (6, 4)\} \]

a. Is \( 4 \, R \, 6 \)?
b. Is \( R \) a function?
c. Does \( R \) have an inverse? If so what is it.
d. Draw the directed graph of \( R \).
e. Is \( R \) reflexive? Explain.
f. Is \( R \) symmetric? Explain.
g. Is \( R \) transitive? Explain.
h. Is \( R \) antisymmetric? Explain.
i. Is \( R \) an equivalence relation? Why or why not?
j. Is \( R \) a partial order relation? Why or why not?

2. The congruence modulo 2 relation \( E \) is defined from \( \mathbb{Z} \) to \( \mathbb{Z} \) as follows:
\[ \forall (x, y) \in E \iff 2 \mid (x - y) \]

a. Is \( 18 \, E \, 94 \)?
b. Is \( E \) a function?
c. Is \( E \) reflexive? Explain.
d. Is \( E \) symmetric? Explain.
e. Is \( E \) transitive? Explain.
f. Is \( E \) antisymmetric? Explain.
g. Is \( E \) an equivalence relation? Why or why not?
h. Is \( E \) a partial order relation? Why or why not?

3. \( D \) is the binary relation defined from \( \mathbb{R} \) to \( \mathbb{R} \) as follows:
\[ \forall (x, y) \in D, \ xDy \iff xy > 0 \]

a. Is \(-3 \, D \, 4 \)?
b. Is \( D \) a function?
c. Is \( D \) reflexive? Explain.
d. Is \( D \) symmetric? Explain.
e. Is \( D \) transitive? Explain.
f. Is \( D \) antisymmetric? Explain.
g. Is \( D \) an equivalence relation? Why or why not?
h. Is \( D \) a partial order relation? Why or why not?

4. \( R \) is the binary relation defined on \( A = \{1, 2, 3, 4, 6, 12\} \) as follows:
\[ \forall x, y \in A, \ xRy \iff x \text{ exactly divides } y \]

a. Is \( R \) a function?
b. Is \( R \) reflexive? Explain.
c. Is \( R \) symmetric? Explain.
d. Is \( R \) transitive? Explain.
e. Is \( R \) antisymmetric? Explain.
f. Is \( R \) an equivalence relation? Why or why not?
g. Is \( R \) a partial order relation? Why or why not?
II. 1. Let $R$ be a relation defined on the set of integers $\mathbb{Z}$ such that
   For all integers $x$ and $y$, $xRy$ iff $4 \mid (x^2 - y^2)$.
   a. Verify that $R$ is an equivalence relation.
   b. Find the equivalence classes of $R$.

2. Let $R$ be a relation defined on the set of real numbers $A$ such that
   $\forall x, y \in A, \ xRy \iff x^2 - y^2 = 0$.
   a. Verify that $R$ is an equivalence relation.
   b. List all the members of $[3]$.
   c. Which of the following equivalence classes are equal?
      $[-41], \left[\frac{-2}{3}\right], \left[\frac{3}{4}\right], \left[\frac{6}{9}\right], [4], \left[\frac{16}{24}\right], [41]$.

III. 1. In baseball’s World Series, the first team to win four games wins the series. Suppose team A wins the first two games. How many ways can the series be completed?

2. a. How many ways can the letters of the word DESIGN be arranged in a row?
   b. How many ways can the letters of the word DESIGN be arranged in a row if the letters SIG remain together in a row?
   c. How many ways can the letters of the word DESIGN be arranged in a row if G and N must remain next to each other as either NG or GN?

IV. True or False

1. If $f$ is $1 - 1$, then $f \circ g$ is $1 - 1$.
2. If $f \circ g$ is $1 - 1$, then $f$ is $1 - 1$.
3. If $f \circ g$ is $1 - 1$, then $g$ is $1 - 1$.
4. If a relation is $1 - 1$ and onto, then it is a function.
5. A relation that is not $1 - 1$ can still have an inverse.
6. If relations $R$ and $S$ are reflexive, then $R \cap S$ is reflexive.
7. The set $\{x \in \mathbb{Q} \mid 1 < x < 2\}$ is countable.
Answers:

I 1. a. no       b. yes       c. \( R^{-1} = \{(6, 2), (2, 4), (4, 6)\} \)
d

II. 1. a. reflexive: \( x^2 - x^2 = 0 \) \( \text{if} \) and \( xRx. \)
symmetric: Let \( 4 | x^2 - y^2 \). Then \( x^2 - y^2 = 4s \) for some integer \( s \). \( y^2 - x^2 = -(x^2 - y^2) = -4s = 4(-s) \) where \(-s\) is an integer, so \( 4 | y^2 - x^2 \) and \( yRx. \)
transitive: Let \( 4 | x^2 - y^2 \) and \( 4 | y^2 - z^2 \). Then \( x^2 - y^2 = 4s \) for some integer \( s \) and \( y^2 - z^2 = 4r \) for some integer \( r \). \( x^2 - z^2 = x^2 - y^2 + y^2 - z^2 = 4s + 4r = 4(r + s) \), \( r + s \) is an integer. Therefore \( 4 | (x^2 - z^2) \) and \( xRz. \)

b. \([0] = \{x| x \text{ is even}\} \quad [1] = \{x| x \text{ is odd}\} \)

2. a.
b. \([3] = \{-3, 3\}\)
c. \([-41] = [41], \quad [2/3] = [6/9] = [16/24]\)

III. 1. 15 ways (by tree diagram)

2. a. 720 ways   b. 24 ways   c. 240 ways