Cardinality

Cardinal numbers: 1, 2, 3, 4, etc.
Ordinal numbers: first, second, third, fourth, etc.
The cardinality of a set is the number of elements in the set.

ex.  \( A = \{a, b, c\} \quad B = \{4, 5, 6\} \)

Sets that have the same number of elements have the same cardinality.

Definition: Let A and B be any sets. A has the same cardinality as B iff there exists a 1–1 correspondence from A to B.

\[ \exists f : A \to B, \quad f \text{ is } 1-1 \text{ and onto} \]

A finite set is one that has no elements or that can be put into a 1–1 correspondence with a set of the form \( \{1, 2, \ldots, n\} \) for some positive integer n.

An infinite set is a nonempty set that can’t be put into a 1–1 correspondence with a set \( \{1, 2, \ldots, n\} \) for any positive integer n.
Theorem 7.6.1
For all sets A, B, C
1. A has the same cardinality as A (reflexive property)

2. If A has the same cardinality as B, then B has the same cardinality as A (symmetric property)

3. If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C. (transitive property)

So A and B have the same cardinality iff A has the same cardinality as B or B has the same cardinality as A.

Infinite sets:

\[ Z^+ = N \] is said to be countably infinite. The cardinality is \( \aleph_0 \) “aleph null”

If a set can be put into a 1 – 1 correspondence with \( Z^+ \), it is countably infinite. If a set is finite or countably infinite, then it is called countable.
ex. Is $\mathbb{Z}$ countable?

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}^+$, 

$$f(n) = 
\begin{cases} 
2n & \text{if } n \in \mathbb{Z}^+ \\
-2n + 1 & \text{if } n \in \mathbb{Z}^{\text{nonpositive}} 
\end{cases}$$
Any subset of any countable set is countable.
ex. Subsets of $\mathbb{Z}$

Let $2\mathbb{Z}$ be the set of even integers.
Let $f : \mathbb{Z} \to 2\mathbb{Z}$, $f(n) = 2n$

ex. Show that $\{3^n \mid n \in \mathbb{Z}\}$ is countable.
ex. positive rational numbers (Q^+)

Do: Show that \( \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\} \) is countable.
Theorem 7.6.2: The set of all real numbers between 0 and 1 $(0 \leq x \leq 1)$ is uncountable.
ex. Show that the sets \{x \mid 2 \leq x \leq 5\} and \{x \mid 0 \leq x \leq 1\} have the same cardinality.
ex. Show that the sets \( \{ x | 0 \leq x \leq 1 \} \) and \( \{ x | -2 \leq x \leq 0 \} \) have the same cardinality.
Do: 1. Set up a $1 - 1$ correspondence to show that the set of multiples of 5 has the same cardinality as the set of multiples of 6.

2. Show that the sets $\{x \mid -1 \leq x \leq 1\}$ and $\{x \mid 1 \leq x \leq 100\}$ have the same cardinality.