Partial Order Relations

Definition: A relation R on a set A is antisymmetric iff
\[ \forall a, b \in A, \text{ if } aRb \text{ and } bRa, \text{ then } a = b. \]
(If \( a \neq b \), then if \( aRb \), \( bRa \).)

ex. \( R = \{(0, 0), (0, 3), (1, 0), (1, 3), (2, 2), (3, 2), (3, 3)\} \)

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ex. $R = \{(a, a), (b, b)\}$

Note: Some relations could be neither symmetric or antisymmetric.

Definition: A partial order relation on a set $A$ is a relation that is reflexive, antisymmetric and transitive.

ex. $A = \mathbb{N}$ $(\mathbb{Z}^+)$ $nRm \iff n \mid m$
ex. Let $A$ be the set of lines in the plane. If $x$ and $y$ are two lines in $A$, then $xRy$ iff $x$ intersects $y$. Is $R$ a partial order relation?

ex. $P$ is the set of all people in the world. If $x$ and $y$ are people, then $xRy$ iff $x$ is no older than $y$ (in days). Is $R$ a partial order relation?
ex. $x R y$ iff $x = y$. Is $R$ a partial order relation?

Do: Determine if the following are partial order relations.

1. Define $R$ on the set of integers $\mathbb{Z}$, $\forall m, n \in \mathbb{Z}$, $m R n \iff$ every prime factor of $m$ is a prime factor of $n$.

2. Define $R$ on $P\{a, b, c\}$,
   $\forall A, B \in P\{a, b, c\}$, $(A) R (B) \iff A \subseteq B$. 
Hasse Diagrams

Start with a directed graph of a relation in which all arrows point upward. Then eliminate:

1. all loops at vertices
2. all arrows whose existence is implied by the transitive property
3. the direction indicators on the arrows.

ex. Let \( S = \{0, 1\} \). \( \forall A, B \subseteq S, \ ARB \iff A \subseteq P(B) \)

\( P(S) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\} \)
ex. (Example 10.5.12) At an automobile assembly plant, the job of assembling an automobile can be broken down into these tasks:
1. Build frame.
2. Install engine, power train components, gas tank.
3. Install brakes, wheels, tires.
4. Install dashboard, floor, seats.
5. Install electrical lines.
6. Install gas lines.
7. Install break lines.
8. Attach body panels to frame.

Certain of these tasks can be carried out at the same time, whereas some cannot be started until other tasks are finished.

<table>
<thead>
<tr>
<th>Task</th>
<th>Immediately Preceding Tasks</th>
<th>Time Need to Perform Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>7 hours</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>6 hours</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3 hours</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>6 hours</td>
</tr>
<tr>
<td>5</td>
<td>2, 3</td>
<td>3 hours</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>1 hour</td>
</tr>
<tr>
<td>7</td>
<td>2, 3</td>
<td>1 hour</td>
</tr>
<tr>
<td>8</td>
<td>4, 5</td>
<td>2 hours</td>
</tr>
<tr>
<td>9</td>
<td>6, 7, 8</td>
<td>5 hours</td>
</tr>
</tbody>
</table>

Let T be the set of all tasks and consider the partial order relation \( \leq \) defined on T as follows:
\( \forall x, y \in T, \; x \leq y \leftrightarrow x = y \) or \( x \) precedes \( y \).

What is the minimum time required to assemble a car?