Relations

A binary relation is a set of ordered pairs.

ex. $S = \{(4, 6), (3, 7), (1, 1), (-2, 2)\}$

Definition: A binary relation of the sets $A$ and $B$ is any subset of $A \times B$.

ex. $A = B = \mathbb{Z}$, $S \subseteq A \times B$

ex. $A = \{-2, 1, 3, 4\}$, $B = \{1, 2, 6, 7\}$, $S \subseteq A \times B$

Also, $T = \{(-2, 1), (4, 7)\}$ is a binary relation of $A$ and $B$.

We use the notation $xRy$ to mean “$x$ is related to $y$”.

$xRy \iff (x, y) \in R$ where $R$ is the set of ordered pairs.

In the above examples

ex. $A = B = \mathbb{Z}$. Define a relation $R = \{\ldots, (-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), \ldots\}$ as $\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}, \ xRy \iff x = y$
Some other relations $x < y$, $x \neq y$, $x | y$

ex. Let $R$ be the relation “is a divisor of”

$R =$

ex. Let $A = B =$ . Define the binary relation $R$ as

$(x, y) \in R \iff |x - y| < 1$

Is $1 R 3$? $0 R 1/2$? $5 R 5$?

List 5 other ordered pairs of $R$. 
Define $R$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$ as given

$(x, y) \in R \iff x^2 + y^2 = 25$.

Is $5R0$?  $5R5$?  $3R4$?

Functions are special relations. A function $F$ from set $A$ to set $B$ is a relation that satisfies two properties:

1. $\forall x \in A, \forall y \in B, (x, y) \in F$. (All of the domain is used.)

2. $\forall x \in A$ and $y$ and $z \in B$, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.
   (Each $x$ is related to only one $y$.)

ex. $A = \{1, 2, 3\}$  $B = \{4, 5, 6\}$

Functions?

$\{(1, 4), (2, 5), (3, 6)\}$

$\{(1, 4), (2, 4), (3, 4)\}$

$\{(1, 4), (1, 5), (2, 6), (3, 5)\}$

$\{(1, 4), (2, 5)\}$
ex. Define $R$ from $\mathbb{Z}$ to $\mathbb{Z}$ as given
$$ (x, y) \in R \iff x^2 + y^2 = 25. $$
Is $R$ a function?

Do: Let $R$ be the relation from $\mathbb{Z}$ to $\mathbb{Z}$,
$$ (x, y) \in \mathbb{Z} \times \mathbb{Z}, (x, y) \in R \iff \text{\textquotedblleft}x \text{ and } y \text{ have no common factor other than 1.\textquotedblright} $$

1. Is $2R7$? $2R8$? $2R2$?

2. Is $R$ a function? Why or why not?
Inverse

Let $R$ be a relation from $A$ to $B$. Define the inverse relation $R^{-1}$ from $B$ to $A$ as

$$R^{-1} = \left\{(y, x) \mid (x, y) \in R \right\}$$

i.e. $(y, x) \in R^{-1}$ if $(x, y) \in R$

ex. $R = \{(1, 4), (2, 5), (3, 6)\}$

$R^{-1} = \{\}$

ex. $R = \{(x, y) \mid x^2 + y^2 = 25\}$

$R^{-1} = \{\}$
Graphs of Relations

The directed graph on a set $A$ depicts a relation from $A$ to $A$. Draw an arrow from each point to each related point.

ex. $A = \{-5, 0, 5\}$ \( x, y \in A \), \( (x, y) \in R \) \( x^2 + y^2 = 25 \).

ex. $A = \{2, 3, 4, 5, 6, 7\}$, $R$ = “no common factors other than 1”