Math 2214, Test 2

1. Solve the initial value problem

\[ y'' - y = e^t, \quad y(0) = y'(0) = 0. \]

The complementary solution is \( y_c = c_1 e^t + c_2 e^{-t}. \) A particular solution has the form \( y_p = Ate^t. \) This leads to \( y'' = A(t + 2)e^t, \) and plugging into the equation yields \( A = 1/2. \) Therefore,

\[ y = \frac{t}{2} e^t + c_1 e^t + c_2 e^{-t}. \]

The initial conditions yield \( c_1 + c_2 = 0, \) \( 1/2 + c_1 - c_2 = 0, \) which leads to \( c_1 = -1/4, c_2 = 1/4. \)

2. A spring is stretched by 4 inches when a weight of 0.5 lbs is hung on it. Determine the natural frequency of the spring. Now the spring is stretched by an additional 2 inches and then released. Determine the motion of the spring. Ignore damping and use \( g=32 \text{ ft/sec}^2. \)

The spring constant (in lbs/ft) is \( k = (1/2)/(1/3) = 3/2, \) and the mass is \( (1/2)/32 = 1/64. \) Therefore the equation is

\[ \frac{y''}{64} + \frac{3}{2} y = 0, \]

or

\[ y'' + 96y = 0. \]

The natural frequency is \( \omega_0 = \sqrt{96} = 4\sqrt{6}. \) The motion of the spring after additional stretching is given by

\[ y = \frac{1}{6} \cos(4\sqrt{6}t). \]

3. Find the general solution of the equation

\[ y'' - y = \frac{1}{1 + e^t}. \]

The complementary solution is as in Problem 1, so we look for the solution in the form \( y = u_1 e^t + u_2 e^{-t}. \) We need to satisfy

\[ u_1'e^t + u_2'e^{-t} = 0, \]

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\[ u_1' e^t - u_2 e^{-t} = \frac{1}{1 + e^t}. \]

This leads to
\[ u_1' = \frac{e^{-t}}{2(1 + e^t)}, \quad u_2' = -\frac{e^t}{2(1 + e^t)}. \]

Integration yields
\[ u_1 = -\frac{1}{2} e^{-t} + \frac{1}{2} \ln(1 + e^{-t}) + c_1, \quad u_2 = -\frac{1}{2} \ln(1 + e^t) + c_2, \]
\[ y = \frac{1}{2} (-1 + e^t \ln(1 + e^{-t}) - e^{-t} \ln(1 + e^t)) + c_1 e^t + c_2 e^{-t}. \]

4. Find the general solution of the equation \( y^{(5)} - y'' = 0. \)

The characteristic equation is \( \lambda^5 - \lambda^2 = \lambda^2(\lambda^3 - 1) = \lambda^2(\lambda - 1)(\lambda^2 + \lambda + 1) = 0, \)
leading to a double root at 0 and simple roots at 1 and \(-1/2 \pm i\sqrt{3}/2.\)

The general solution is
\[ y = c_1 + c_2 t + c_3 e^t + c_4 e^{-t/2} \cos(\sqrt{3}t/2) + c_5 e^{-t/2} \sin(\sqrt{3}t/2). \]