Math 2214, Test 2

1. Solve the initial value problem

\[ y'' - y = e^t, \quad y(0) = 0, \quad y'(0) = 0. \]

The complementary solution is

\[ y_c = c_1 e^t + c_2 e^{-t}. \]

A particular solution has the form \( y_p = Ate^t \). Plugging this into the equation leads to \( A = 1/2 \). The general solution is

\[ y = \frac{1}{2} te^t + c_1 e^t + c_2 e^{-t}. \]

We have \( y(0) = c_1 + c_2 = 0, \ y'(0) = 1/2 + c_1 - c_2 = 0 \). This leads to \( c_1 = -1/4, c_2 = 1/4 \).

2. A spring is stretched by 4 inches when a weight of 0.5 lbs is hung on it. Determine the natural frequency of the spring. Now the spring is stretched by an additional 2 inches and then released. Determine the motion of the spring. Ignore damping and use g=32 ft/sec².

The spring constant is \( 0.5/(1/3) = 1.5 \) lb/ft, and the mass is \( 0.5/32 = 1/64 \) lb sec²/ft. The equation of motion is

\[ \frac{1}{64} y'' + \frac{3}{2} y = 0, \]

which has solutions

\[ y = c_1 \cos(4\sqrt{6}t) + c_2 \sin(4\sqrt{6}t). \]

The initial conditions lead to \( c_1 = 1/6, c_2 = 0 \).

3. Find the solution of the initial value problem

\[ y'' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -10. \]

Determine the period, amplitude and phase.

The general solution is

\[ y = c_1 \cos(5t) + c_2 \sin(5t). \]
The initial conditions lead to $c_1 = 2$, $c_2 = -2$. The period is $2\pi/5$, the amplitude is $2\sqrt{2}$ and the phase is $-\pi/4$, i.e.

$$y = 2\sqrt{2}\cos(5t + \pi/4).$$

4. The functions $y_1 = \cos t$ and $y_2 = t\cos t$ are solutions of the equation

$$y'' + 2(\tan t)y' + (1 + 2\tan^2 t)y = 0.$$

Find a particular solution of

$$y'' + 2(\tan t)y' + (1 + 2\tan^2 t)y = \cos t.$$

We look for $y = u_1 \cos t + u_2 t \cos t$. We need to solve

$$u_1' \cos t + u_2' t \cos t = 0,$$

$$-u_1' \sin t - u_2' t \sin t + u_2' \cos t = \cos t.$$

This leads to the solution $u_2' = 1$, $u_1' = -t$, i.e. $u_1 = -t^2/2 + c_1$, $u_2 = t + c_2$. The solution is

$$y = u_1 \cos t + u_2 t \cos t = \frac{1}{2}t^2 \cos t + c_1 \cos t + c_2 t \cos t.$$