Solutions to Test 1:

1) (8pts) Let \( H : \) doing homework, \( C : \) passing this course

- a) \( H \rightarrow C \) where \( H \) is the sufficient condition and \( C \) is the necessary condition
- b) \( C \rightarrow H \) where \( C \) is the sufficient condition and \( H \) is the necessary condition
- c) \( C \rightarrow H \) (same as b)
- d) \( \sim H \rightarrow \sim C \equiv (C \rightarrow H) \) by the contrapositive

Therefore b), c) and d) are equivalent.

2) (16 pts)

a) If \( n \) is odd then there is an integer \( k \) that will give \( n = 2k + 1 \). (True)

b) There is an odd integer \( n = 2k + 1 \) where \( k \) can be any integer value. (False)

c) There is at least one integer \( k \) that can be used to represent any odd integer \( n = 2k + 1 \). (False)

d) \( \sim \exists k \in \mathbb{Z}, \forall n \in \mathbb{Z}, n = 2k + 1 \equiv \forall k \in \mathbb{Z}, \exists n \in \mathbb{Z}, n \neq 2k + 1 \)

Given any integer \( k \), there is an integer \( n \) and \( n \neq 2k + 1 \). (True)

3) (12pts) We are given that \( \sim k \) is true, therefore since \( (m \rightarrow k) \equiv (\sim k \rightarrow \sim m) \) is a true statement, we have that \( \sim m \) is true and \( m \) is false. The statement \( p \lor m \) indicates that \( p \) is true. Next we need to consider the statement \( \sim m \rightarrow (y \land q) \) which forces \( (y \land q) \) to be true since \( \sim m \) is true. Therefore both \( y \) and \( q \) are true. We now have that \( p \) and \( q \) are both true and \( (p \land q) \rightarrow w \), this will give us that valid conclusion \( w \).

4) (6pts) Given that \( b \rightarrow (m \lor w) \) is false, this means that \( b \) is true and \( (m \lor w) \) is false. We know that both \( m \) and \( w \) have to be false.

a) The statement \( m \) is false
b) \( w \rightarrow b \) is true since \( w \) is false and \( b \) is true.

5) (6pts) By the Quotient Remainder Theorem, since we have \( x \mod 6 = 4 \) and \( y \mod 6 = 5 \), then \( x = 6q + 4 \) and \( y = 6d + 5 \).

Then \( x \times y = (6q + 4)(6d + 5) = 36qd + 30q + 24d + 20 = 6(6qd + 5q + 4d + 3) + 2 \)

By definition of Quotient Remainder Theorem we have \( xy \mod 6 = 2 \)

6) (6pts) Let \( B \) be the set of birds in the wildlife shelter and \( F \) is the set of foxes in the wildlife shelter and let \( P(x, y) \) be the statement \( x \) and \( y \) in the shelter

\( \exists x \in B, \exists y \in F \mid P(x, y) \)

The truth set for this statement is two eagles, four hawks, one owl and one hokie bird in \( B \) and anyone of three foxes in \( F \).
7)  (8pt)
\[ \sim [m \to (s \land r)] \land \sim s \equiv \]
\[ \sim [\sim m \lor (s \land r)] \land \sim s \equiv \]
\[ m \land \sim (s \land r) \land \sim s \equiv \]
\[ m \land (\sim s \lor \sim r) \land \sim s \equiv \]
\[ m \land \sim s \]

8)  (12pts) Theorem: The sum of a rational number and an irrational number is always irrational.

Proof by contradiction:
Assume a rational number and an irrational number is rational: Let \( r \) be a rational number and \( r = a/b \) where \( a, b \neq 0 \) and \( k \) is an irrational number. Assume the sum is rational. Therefore \( r + k = a/b + k = c/d \) where \( c, d \neq 0 \).

Consider \[ a/b + k = c/d \]
\[ k = c/d - a/b \]
since we know that the difference rational number then \( k \) is a rational number, but this is a contradiction to the given property that \( k \) is irrational. Therefore the original statement is true.

9) (12pt) Theorem: If \( a \) does not divide \( b^2 \), then \( a \) does not divide \( b \).

Proof by contrapositive: If \( a \) divides \( b \) then \( a \) divides \( b^2 \).

Since \( a \) divides \( b \), by definition of divisible we have \( b = aq \)
Therefore \( bb = baq \) and \( b^2 = a(bq) = ad \)
This gives us that \( b^2 \) is divisible by \( a \).
We have proved the contrapositive is true and therefore the original statement is true since it is equivalent to its contrapositive.

10) (12pt) Theorem: If \( n \) is positive prime integer \( > 2 \) then \( n \) is odd.

Use proof by contradiction: Assume that \( n \) is even. Then \( n = 2k \) by definition of even and \( n \) can be divided by the integers 1, \( n \), 2, and \( k \). The contradicts the definition of prime which states that a prime number can only be divided by 1 or itself. Therefore all prime numbers \( > 2 \) must be odd.