REDUCED ROW ECHelon MATRICES:

The following matrices are in reduced row echelon form:

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -3
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 4 & 0 & 0 & -3 \\
0 & 0 & 1 & 6 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 4 & 0
\end{bmatrix}
\]

Adding a vertical line to indicate an Augmented Coefficient has no effect on whether a matrix is reduced or not. The same matrices written as augmented coefficient matrices are also reduced.

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -3
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 4 & 0 & 0 & -3 \\
0 & 0 & 1 & 6 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 4 & 0
\end{bmatrix}
\]

The following matrices are NOT in reduced row echelon form. Why not (which rule do they violate)?

(a) \[
\begin{bmatrix}
0 & 1 & -2 \\
1 & 0 & 3
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & 2 & -2 & 3 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & -2
\end{bmatrix}
\]
(d) \[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 2 & 0 & 3
\end{bmatrix}
\]

**SOLUTIONS TO SYSTEMS OF EQUATIONS:**

If an augmented coefficient matrix is in reduced row echelon form, the following are true for real numbers a, b, c, d, and p with p \(\neq 0\):

1. If the reduced matrix has a form similar to that in the following 3x3 matrix, the system of equations has a unique solution. We say the system of equations is consistent and independent.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\]

If the left portion of the matrix (the coefficient matrix) represents the coefficients to the variables x, y, and z, the matrix represents the system of equations \(x = a\), \(y = b\), and \(z = c\).

2. If the reduced matrix has a form similar to that in the following 3x3 matrix, the system of equations has infinitely many solutions. The system of equations is consistent and dependent.

\[
\begin{bmatrix}
1 & 0 & c \\
0 & 1 & d \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
0
\end{bmatrix}
\]

If the left portion of the matrix (the coefficient matrix) represents the coefficients to the variables x, y, and z, the solution to the system of equations is given by: \(x = a - ct\), \(y = b - dt\), and \(z = t\), for some parameter t.

3. If the reduced matrix has a form similar to that in the following 3x3 matrix, the system of equations is inconsistent and has no solution.

\[
\begin{bmatrix}
1 & 0 & c \\
0 & 1 & d \\
0 & 0 & p
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
p
\end{bmatrix}
\]

IMPORTANT NOTE: In general, a system of equations with more variables than equations (that is, the coefficient matrix has more columns than rows) cannot have a unique solution. It must have either no solution or infinitely many solutions.
AN APPLICATION PROBLEM:
A casting company produces three different bronze sculptures. The casting department has available a maximum of 350 labor-hours per week, and the finishing department has a maximum of 150 labor-hours available per week. Sculpture A requires 30 hours for casting and 10 hours for finishing; sculpture B requires 10 hours for casting and 10 hours for finishing; and sculpture C requires 10 hours for casting and 30 hours for finishing. If the plant is to operate at maximum capacity, how many of each sculpture should be produced each week?

Solution:
First, we summarize the relevant manufacturing data in a table (note values form a matrix with row and column headings):

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Maximum labor-hours available per week</th>
</tr>
</thead>
<tbody>
<tr>
<td>Casting Department</td>
<td>30</td>
<td>10</td>
<td>10</td>
<td>350</td>
</tr>
<tr>
<td>Finishing Department</td>
<td>10</td>
<td>10</td>
<td>30</td>
<td>150</td>
</tr>
</tbody>
</table>

Let  \( x_1 \) = Number of sculpture A produced per week
\( x_2 \) = Number of sculpture B produced per week
\( x_3 \) = Number of sculpture C produced per week.

Then
\[
30x_1 + 10x_2 + 10x_3 = 350 \quad \text{(Casting department)} \\
10x_1 + 10x_2 + 30x_3 = 150 \quad \text{(Finishing department)}
\]

Now we can form the augmented coefficient matrix of the system and solve by using Gauss-Jordan elimination:

\[
\begin{bmatrix}
30 & 10 & 10 & 350 \\
10 & 10 & 30 & 150 \\
\end{bmatrix}
\]

(Simplify each row)

\[
\begin{bmatrix}
3 & 1 & 1 & 35 \\
1 & 1 & 3 & 15 \\
\end{bmatrix}
\]

(Interchange row 1 and row 2 to get leading 1 in row 1)

\[
\begin{bmatrix}
1 & 1 & 3 & 15 \\
3 & 1 & 1 & 35 \\
\end{bmatrix}
\]

(Get a zero in row 2, column 1)

\[
\begin{bmatrix}
1 & 1 & 3 & 15 \\
0 & -2 & -8 & -10 \\
\end{bmatrix}
\]

(Get leading 1 in row 2)

\[
\begin{bmatrix}
1 & 1 & 3 & 15 \\
0 & 1 & 4 & 5 \\
\end{bmatrix}
\]

(Get zero in row 1 in same column as leading 1 in row 2)

\[
\begin{bmatrix}
1 & 0 & -1 & 10 \\
0 & 1 & 4 & 5 \\
\end{bmatrix}
\]

(Matrix is now in reduced row echelon form)

Now, from above matrix, \( x_1 - x_3 = 10 \) and \( x_2 + 4x_3 = 5 \), or \( x_1 = 10 + x_3 \) and \( x_2 = 5 - 4x_3 \).

Let \( x_3 = t \). Then for any real number \( t \), \( x_1 = t + 10 \)

\[
x_2 = -4t + 5 \\
x_3 = t
\]

is a solution – or is it??????

We cannot produce a negative number of sculptures. If we also assume that we cannot produce a fractional number of sculptures, then \( t \) must be a nonnegative whole number. And because of the middle equation (\( x_2 = -4t + 5 \)), \( t \) can only assume the values 0 and 1. Thus, for \( t = 0 \), we have \( x_1 = 10, x_2 = 5, x_3 = 0 \); and for \( t = 1 \), we have \( x_1 = 11, x_2 = 1, \) and \( x_3 = 1 \). These are the only possible production schedules that can utilize the full capacity of the plant.