Math 1526
Consumer and Producer Surplus

Scenario: In the grocery store, I find that two-liter sodas are on sale for 89¢. This is good news for me, because I was prepared to pay $1.29 for them. The store manager is happy to see me loading them into my cart, because she knows the store’s wholesale cost is just 58¢ a bottle.

I’m thinking: “Lucky me! I’m paying 89 cents for a $1.29 item, so I’m coming out 40 cents ahead on each bottle!”

She’s thinking: “Lucky me! I’m selling these 58-cent sodas for 89 cents, so I’m coming out 31 cents ahead on each bottle!”

Both parties feel like they’re coming out ahead on the transaction. The fact that a transaction can be beneficial to both parties is a fundamental principal of economics. The term surplus is used to describe how much each party gains from the transaction.

Exercise: The store manager is the producer in this scenario, because she’s making the soda available to the consumer. But earlier, she was a consumer, buying the products wholesale from the manufacturer. How might that scenario play out?

Before we can discuss surplus in detail, we first need to study equilibrium.

Example 1: Suppose we’re given the demand function \( p = -50q + 2000 \). This demand function is graphed to the right. From the graph, we can determine how many consumers would be willing to pay a given price for this item. For example, only twenty consumers would be willing to pay $1000 for it. Because fewer consumers would be willing to pay a higher price, a demand function always has a negative slope.

Now, suppose we had the supply function \( p = 10q + 500 \). This supply function is graphed to the left. From this graph, we can determine how many units a producer would supply at a given price. For example, the producer would supply 10 units at $600 each. Of course, the producer would rather sell them at an even higher price, and would be willing to supply more products at that price. So a supply function always has a positive slope.
If we plot the supply function and the demand function on the same axis, we see that they intersect at a point. This point is called the equilibrium point. It represents the place where demand exactly equals supply. (In theory, the very last customer walks in just in time to pull the last item off the shelf!)

Because the equilibrium point is a point, it has two coordinates—the equilibrium quantity $E_q$, and the equilibrium price $E_p$.

**Exercise:** Estimate from the graph the coordinates of the equilibrium point.

### Solving for the equilibrium quantity and price

To find the equilibrium quantity, set the supply and demand equations equal to each other:

\[
\begin{align*}
supply &= demand \\
10q + 500 &= -50q + 2000 \\
60q &= 1500 \\
q &= 25
\end{align*}
\]

So we know that $E_q = 25$.

Once you have the equilibrium quantity, you can find the equilibrium price by plugging in to either the supply equation or the demand equation. Both functions will give you the same answer, and you can use the second function to check your work.

Using the supply equation:

\[
\begin{align*}
p &= 10q + 500 \\
p &= 10(25) + 500 \\
p &= 250 + 500 \\
p &= $750
\end{align*}
\]

Using the demand equation:

\[
\begin{align*}
p &= -50q + 2000 \\
p &= -50(25) + 2000 \\
p &= -1250 + 2000 \\
p &= $750
\end{align*}
\]

Since both agree, we are confident that $E_p = $750. The equilibrium price is also sometimes called fair market value. The equilibrium point is at (25,750). Does this match the estimate you eyeballed from the graph?
Preparing to find surplus

Draw a line connecting the equilibrium point to the price axis. This line will extend from a quantity of zero all the way to the equilibrium quantity:

Let’s interpret the meaning of this line at a particular quantity: When \( q = 10 \), the demand equation is at \( p = $1500 \), and the supply equation is at \( p = $600 \). The equilibrium price is $750.

The consumers are thinking: “We would have bought 10 units for $1500, but instead we got them for the equilibrium price of $750! What a deal! We came out $750 ahead on each of these items!”

The producers are thinking: “We would have sold 10 units for $600, but instead we sold them for the equilibrium price of $750. What a deal! We made an extra $150 on each of these items!”

This interpretation applies to all quantities up to the equilibrium point.

Consumer Surplus

Consumer surplus is the total amount by which the consumers came out ahead. It’s equal to the area between equilibrium and demand. (Remember, consumers create the demand for the product!) It’s the shaded in area on this graph:
Producer Surplus

Producer surplus is the total amount by which the producers came out ahead. It’s equal to the area between equilibrium and supply. (Remember, producers create the supply of the product!) It’s the shaded in area on this graph:

![Example 1, Producer surplus](image)

**Solving for Consumer and Producer Surplus**

Both areas can be found using a definite integral. The form the integral takes is:

\[
\int_{x_{\text{left}}}^{x_{\text{right}}} (\text{upper function}) - (\text{lower function}) \, dx
\]

Look again at the shaded area for Consumer Surplus. The left edge of the triangle has an \(x\)-coordinate of 0, and the right edge is our equilibrium point, which has an \(x\)-coordinate of 25. The top of the triangle is the demand equation \(p = -50q + 2000\), and the bottom of the triangle is our constant equilibrium price, 750. So,

\[
\text{Consumer Surplus} = \int_0^{25} (-50q + 2000) - 750 \, dq
\]

\[
= \left[ -25q^2 + 2000q - 750q \right]_0^{25}
\]

\[
= \left[ -25(25)^2 + 2000(25) - 750(25) \right] - \left[ -25(0)^2 + 2000(0) - 750(0) \right]
\]

\[
= $15,625
\]

Producer Surplus is found the same way: The left edge and the right edge are still at 0 and 25, but now the top of the triangle is our equilibrium price, and the bottom of the triangle is our supply equation \(p = 10q + 500\). So,

\[
\text{Producer Surplus} = \int_0^{25} (750) - (10q + 500) \, dq
\]

\[
= \left[ 750q - 5q^2 - 500q \right]_0^{25}
\]

\[
\leftarrow \text{remember to distribute the } - \text{ sign!}
\]

\[
= \left[ 750(25) - 5(25)^2 - 500(25) \right] - \left[ 750(0) - 5(0)^2 - 500(0) \right]
\]

\[
= $3125
\]
**Consumer and Producer Surplus—Another Example**

**Example 2:** Suppose this time, the demand function is given by \( q = -0.5p + 70 \) and the supply function is given by \( q = 0.7p - 50 \). This example is different from example 1, because price is on the \( x \)-axis, and quantity is on the \( y \)-axis. This makes the problem a little more complicated—but just a little!

\[
\begin{align*}
\text{Example 2: Demand and Supply} \\
\begin{array}{c}
\text{demand} \\
\text{supply}
\end{array}
\end{align*}
\]

To find surplus, we’ll follow the same steps as before. These steps were marked with a blue dot in Example 1:

- Solve for the equilibrium quantity and price
- Draw a line from the equilibrium point to the price axis
- Set up the definite integrals needed and solve them

### Solving for the equilibrium quantity and price

As before, we set the supply and demand equations equal to each other. This time, however, we’ll be solving for \( p \), instead of \( q \):

\[
\begin{align*}
\text{supply} &= \text{demand} \\
0.7p - 50 &= -0.5p + 70 \\
1.2p &= 120 \\
p &= 100
\end{align*}
\]

So we know that \( E_p = \$100 \). To find \( E_q \), we could use either the supply or the demand equation. Again, both will give the same answer:

\[
\begin{align*}
\text{supply:} & \quad \text{demand:} \\
q &= 0.7(100) - 50 \\
q &= -0.5(100) + 70 \\
q &= 70 - 50 \\
q &= -50 + 70 \\
q &= 20 \\
q &= 20
\end{align*}
\]

Both solutions agree, so we can be sure that \( E_q = 20 \text{ units} \).
Draw a line from the equilibrium point to the price axis

This time, our line will be vertical instead of horizontal:

Nevertheless, Consumer Surplus is still the area between the equilibrium and demand, and Producer Surplus is still the area between equilibrium and supply. These areas are shown in the graphs below:

Set up the integrals needed and solve them

The definite integral needed to solve for these areas will still take the form:

\[ \int_{x\text{-coordinate of left edge}}^{x\text{-coordinate of right edge}} (\text{upper function}) - (\text{lower function}) \, dx \]

**Consumer Surplus**

The left edge of Consumer Surplus is the equilibrium line. And the x-coordinate of that line is our equilibrium price, or $100. The right edge is the point where the demand function crosses the x-axis. To find this point, we set the demand function equal to zero and solve:

\[
\text{demand} = 0 \\
-0.5p + 70 = 0 \\
-0.5p = -70 \\
p = 140
\]

We didn’t have to do this step in Example 1. We only have to solve to find our bounds when \( p \) is on the x-axis.
So the bounds of our integral will be at $100$ and $140$. The upper function is our demand equation, and our lower function is the $x$-axis—or in other words, 0! This leads to the integral:

$$\int_{100}^{140} (-0.5p + 70) - (0) \, dp$$

We can solve this integral readily:

$$\int_{100}^{140} (-0.5p + 70) - (0) \, dp = \left[ \frac{-0.5p^2}{2} + 70p + C \right]_{100}^{140}
= \left[ \frac{-0.5(140)^2}{2} + 70(140) + C \right] - \left[ \frac{-0.5(100)^2}{2} + 70(100) + C \right]
= \$400$$

**Producer Surplus**

The left edge of Producer Surplus is the point where the supply function crosses the $x$-axis, and so to find this point, we set the supply function equal to zero and solve:

\[
supply = 0 \\
0.7p - 50 = 0 \\
0.7p = 50 \\
p = \$$71.43$$

Not a pretty number, but then again, not every problem works out nicely. Fortunately, the right edge is easy: it's the equilibrium line, and the $x$-coordinate of that line is $100$. So the bounds of our integral will be $71.43$ and $100$. The upper function this time is our supply equation, and the lower function is zero. This leads to the integral:

$$\int_{71.43}^{100} (0.7p - 50) - (0) \, dp$$

We can solve this integral as well:

$$\int_{71.43}^{100} (0.7p - 50) - (0) \, dp = \left[ \frac{0.7p^2}{2} - 50p + C \right]_{71.43}^{100}
= \left[ \frac{0.7(100)^2}{2} - 50(100) + C \right] - \left[ \frac{0.7(71.43)^2}{2} - 50(71.43) + C \right]
= \$$285.71$$

So Consumer Surplus is $400$, and Producer Surplus is $285.71$. 
Consumer and Producer Surplus—Other Types of Examples

Both of the examples we’ve seen have been straightforward. Here we briefly discuss two more difficult scenarios:

Example 3: It’s possible that you don’t have equations to work with at all, but only data. What if you were faced with the following scenario:

<table>
<thead>
<tr>
<th>q</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>21</th>
<th>24</th>
<th>27</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>supply</td>
<td>$5.21</td>
<td>$6.24</td>
<td>$7.41</td>
<td>$8.50</td>
<td>$10.07</td>
<td>$12.39</td>
<td>$14.61</td>
<td>$17.80</td>
<td>$20.20</td>
<td>$24.83</td>
<td>$29.94</td>
</tr>
<tr>
<td>demand</td>
<td>$40.00</td>
<td>$39.48</td>
<td>$38.65</td>
<td>$34.10</td>
<td>$29.43</td>
<td>$24.41</td>
<td>$20.39</td>
<td>$17.60</td>
<td>$14.21</td>
<td>$10.36</td>
<td>$4.97</td>
</tr>
</tbody>
</table>

In a case like this, you couldn’t solve for the equilibrium price and quantity. But you could tell from the table that the equilibrium quantity was close to 21 units and that the equilibrium price was somewhere between $17.60 and $17.80—so we’ll split the difference and guess that it’s $17.70.

Once we know the equilibrium point, we’d like to find the surplus, but we can’t do that using a definite integral because we don’t have a formula to work with. We can, however, use one of our approximation methods (such as Riemann Sums, the Trapezoid Rule, or Simpson’s Rule) to estimate surplus. Remember that the heights in these problems won’t be the numbers directly from the table, but the difference between those prices and the equilibrium price.

Here’s how we might find Consumer Surplus using the Trapezoid Rule:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>quantity</td>
<td>demand</td>
<td>$E_p$</td>
<td>height (demand-$E_p$)</td>
<td>multiplier</td>
<td>product (height x multiplier)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>40</td>
<td>17.7</td>
<td>22.3</td>
<td>1</td>
<td>22.3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>39.48</td>
<td>17.7</td>
<td>21.78</td>
<td>2</td>
<td>43.56</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>38.65</td>
<td>17.7</td>
<td>20.95</td>
<td>2</td>
<td>41.9</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>34.1</td>
<td>17.7</td>
<td>16.4</td>
<td>2</td>
<td>32.8</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>29.43</td>
<td>17.7</td>
<td>11.73</td>
<td>2</td>
<td>23.46</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>24.41</td>
<td>17.7</td>
<td>6.71</td>
<td>2</td>
<td>13.42</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
<td>20.39</td>
<td>17.7</td>
<td>2.69</td>
<td>2</td>
<td>5.38</td>
</tr>
<tr>
<td>9</td>
<td>21</td>
<td>17.6</td>
<td>17.7</td>
<td>-0.1</td>
<td>1</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

Sum: 182.72
Sum x 3 / 2 = Total Area: 274.08

Exercise: Why can’t we use Simpson’s Rule to find this area?

Exercise: Estimate Producer Surplus using the Trapezoid Rule.
Example 4: It’s possible that the functions you’re given could be more difficult to work with. In this example, the demand function is \( p = 100e^{-0.03q} \) and the supply function is \( p = 20e^{0.05q} \). The graph is shown here:

The process for solving an example like this hasn’t changed, but the calculations will be messier.

The equation we’ll have to solve to find our equilibrium quantity is:

\[
\text{supply} = \text{demand} \\
20e^{0.05q} = 100e^{-0.03q} \\
e^{0.05q} = 5e^{-0.03q} \\
e^{0.08q} = 5 \\
0.08q = \ln 5 \\
q = \frac{\ln 5}{0.08} \\
q = 20.12
\]

Finding the equilibrium price involves plugging this value into either the supply equation or the demand equation. For example, using the demand equation, we get:

\[
p = 100e^{-0.03(20.12)} = 54.69
\]

After we’ve done this, we’ll have to set up both definite integrals:

\[
\text{Consumer Surplus} = \int_{0}^{20.12} 100e^{-0.03q} - 54.69 \, dq
\]

\[
= \left[ \frac{100e^{-0.03q}}{-0.03} - 54.69q + C \right]_{0}^{20.12}
\]

\[
= \left[ \frac{100e^{-0.03(20.12)}}{-0.03} - 54.69(20.12) + C \right] - \left[ \frac{100e^{-0.03(0)}}{-0.03} - 54.69(0) + C \right]
\]

\[
= (-2923.36 + C) - (-3333.33 + C)
\]

\[
= 409.97
\]

Exercise: Set up and solve to find Producer Surplus.
Significance of Consumer and Producer Surplus

When Consumer Surplus is high, this means that consumers have more money ‘left over’ to spend than they were expecting. They were prepared to pay higher prices for items they needed, and because they didn’t have to, they have money left over. Some of the effects of a high consumer surplus are:

- Lower crime rate
- Increased sales of non-necessities: vacation travel goes up, movie attendance goes up, etc.

When Producer Surplus is high, this means that producers have more profits than they were expecting. They were able to sell their products for a higher price than they were willing to sell them for. Some of the effects of a high producer surplus are:

- More money invested in product research (resulting in better, safer, more efficient products.)
- More money for company growth and expansion (resulting in more jobs and lower unemployment.)