Problem 1: Given the function \( f(x) = e^x \) defined on the Real Numbers.

a) Using the definition, determine if \( f(x) \) one to one?

b) Using the definition, determine if \( f(x) \) onto?

c) Does \( f(x) \) have an inverse function? If not give explanation and make modifications so that an inverse function exists.

Solutions: (20pts)

To show that \( f(x) \) is one to one we must show that if \( \forall a_1 \) and \( a_2 \) in the domain, \( f(a_1) = f(a_2) \) then \( a_1 = a_2 \).

Suppose that \( f(a_1) = f(a_2) \), then we have that \( e^{a_1} = e^{a_2} \). Therefore \( f(x) \) is one to one.

To show that \( f(x) \) is onto we must show that \( \forall b \in \text{co-domain}, \exists a \in \text{domain} \) so that \( f(a) = b \).

Consider:

\( e^b = a \)

\( \ln e^b = \ln a \)

b \ln e = \ln a

\( b = \ln a \)

Therefore \( f(x) \) is not onto since undefined for \( x \leq 0 \).

\( f(x) \) does not have an inverse unless the domain of the inverse function is restricted to positive real numbers only and \( f^{-1}(x) = \ln x \).

Problem 2: Let \( R \) be a relation on a set \( A = \{a, b, c, d\} \) defined by

\( R = \{(a, b), (b, c), (b, a), (c, c), (c, d)\} \)

Make the necessary minimal deletions or additions to make \( R \) anti-symmetric?

Solutions: (6pts) omit (b,a)

Problem 3: Cardinality can be expressed as a relation acting on sets. Given sets \( A \) and \( B \), \( R \) is if and only if they have the same cardinality. Prove that \( R \) is an equivalence relation on all sets. Remember: Two finite sets have the same cardinality if and only if there is a bijection from one set to the other.

Solution (20pts)

Proof: In order to prove that \( R \) is an equivalence relation, we will show that \( R \) is reflexive, symmetric and transitive.

Reflexive: We must verify that \( ARA \). In order to this I must produce a bijection \( F: A \rightarrow B \). Consider the identity mapping \( F(a_i) \rightarrow a_i, i \in N \). This mapping is clearly both one to one and onto and is therefore a bijection.

Symmetric: If \( ARB \), then we must show that \( BRA \). It is given that there is a bijection \( F \) from \( A \) to \( B \). Since \( F \) is a bijection then the inverse exist, \( F^{-1}B \rightarrow A \) and \( F^{-1} \) is also a bijection. Therefore \( BRA \), and \( R \) is symmetric.
Transitive: If ARB and BRC, we need to show that ARC. We are given a bijection $F : A \to B$ and $G : B \to C$. Since F and G are both bijections, then the composition function $G \circ F$ is also a bijection and maps A to C. Therefore ARC and R is Transitive since R is reflexive, symmetric and transitive, R is an Equivalence relation. **How does R partition the set of all sets??**

**Problem 4:** If f(x) and g(x) are both bijections, does the inverse of the composition function $(g \circ f)$ exist? Justify your answer by quoting appropriate theorems, definitions and/or properties.

**Solutions:** (10pts) Since f(x) and g(x) are both bijections, then we know that the compositions are also bijections. Therefore since $(g \circ f)(x)$ is a bijection, then we know that the inverse must exist which is $(g \circ f)^{-1}(x)$.

**Problem 5:**
Given the following Hasse Diagram, Find the relation R that describes the graph.

**Solution:** (6pts) $R = \{(a,a),(e,e),(b,b),(c,c),(d,d),(a,e),(a,c),(a,d),(b,c),(b,d)\}$

**Problem 6:**
If $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ with $f : A \to B$, $g : B \to A$

$G(f) = \{(1,b),(2,e),(3,d),(4,b)\}$

$G(g) = \{(a,2),(b,1),(c,3),(d,1),(e,4)\}$

a) Find $G(f \circ g)$

b) Is $G(f \circ g)$ a function? If so, state the domain and range of $G(f \circ g)$: $B \to B$

**Solution:** (10pts) $G(f \circ g) = \{(a,e),(b,b),(c,d),(d,b),(e,b)\}$

**Problem 7:** Let R be a relation on Sets defined by $A R B$ if and only if $A \subseteq B$. Given that R is a partial order relation. Using complete sentences in your write up, verify that R is anti-symmetric. (Do not try to also verify the reflexive or transitive properties)

**Solution:** (10pts)
The definition of symmetric is as follows: If ARB AND BRA, then A=B. Since we are given that ARB and BRA, we have that $A \subseteq B$ and $B \subseteq A$. Therefore, by definition of equal sets, we know that $A = B$ and R is anti-symmetric.

**Problem 8:** The functions $f(x) = x^2 + 4x + 4$ and $g(x) = x^2$ are defined on the positive real numbers. Verify that f(x) is Big O g(x) by demonstrating that it fits the definition. You do not need to prove anything.

**Solution:** (8 pts) Definition of BigO notation is as follows: If F(x) is Big O G(x), then $|F(x)| \leq C|G(x)|$ where there exist $C > 0$, $K > 0$, $\forall x > K$

We need to find a value for C and a value for K.

Consider $f(x) = x^2 + 4x + 4 \leq x^2 + 4x^2 + 4x^2 = 9x^2$ so let $C = 9$.

Set $x^2 + 4x + 4 = 9x^2$ and solve for the point of intersection which is $x = 1$. So let $K = 1$. 

Problem 9: How many people must be in one room to guarantee that at least 3 people have the same last initial? Answer: (6 pts) 53 people

Problem 10: The congruence relation \( a \equiv b \mod d \) is an equivalence relation on the integers. Therefore it partitions the integers. Show how the integers would be partitioned. Solutions: (4pts) \( \mathbb{Z} = [0] \cup [1] \cup [2] \cup ... \cup [d - 1] \)