Math 2534_Solutions  Homework on Functions

Problem 1:
   Explain the mistake in the following proof:
   Theorem: If \( f(x) = 4x + 3 \) for all integers, Then \( f(x) \) is one to one.
   Proof: Suppose any integer \( x \) is given. Then by definition of \( f \), there is only one possible value for \( f(x) \), namely \( 4x + 3 \). Hence \( f(x) \) is one to one.
   Solution:
   This proof establishes only that \( f(x) \) is a function which is not the same as being one to one.

Problem 2:
   Define If \( A = \{a, b, c\} \), then define \( F : P(A) \rightarrow \mathbb{Z} \) as follows: For all subsets \( S \) in \( P(A) \), \( F(S) = n(S) \) (ie. the number of elements in \( S \))
   a) Is \( F \) one to one? Prove or give a counter example
   b) Is \( F \) onto? Prove or give a counter example
   Solution: \( F \) is defined as follows: \( F(\emptyset) = 0 \), \( F(\{a\}) = F(\{b\}) = F(\{c\}) = 1 \), \( F(\{a,b\}) = F(\{b,c\}) = F(\{a,c\}) = 2 \), \( F(\{a,b,c\}) = 3 \). Since the Range is contained in the co-domain, \( F \) is not onto and \( F \) is not one to one since more than one element in the domain maps to the same element in the Co-domain.

Problem 3:
   Let \( A \) and \( B \) be finite sets and \( n(A) = n(B) \). If \( f \) maps \( A \) to \( B \), then \( f(x) \) is one to one if and only if \( f(x) \) is onto.
   Solution: Proof:
   Part A: If \( f(x) \) is one to one then \( f(x) \) is onto.
   Proof by contradiction: Assume that \( f(x) \) is not onto. Then there is an element in the Co-domain that is not in the image of \( f(x) \). Since \( n(A) = n(B) \), there are at least two elements in \( A \) that map to the same element in \( B \). This contradicts that \( f(x) \) is one to one. Therefore \( f(x) \) must be onto.

Part B: If \( f(x) \) is onto, then \( f(x) \) is one to one.
   Proof: The image of \( A \) under \( f \) is \( f(A) \). Since \( f(x) \) is onto then \( n(f(A)) = n(B) = n(A) \). So \( f \) is also one to one.
   Therefore \( f(x) \) is one to one if and only if \( f(x) \) is onto.
Problem 4:
Determine if the following is true or false. Justify your conclusion.

a) Given that \( f: A \rightarrow B \) and \( f^{-1} \) exists, then for all subsets \( S \) of \( B \)
\[ f(f^{-1}(S)) \subseteq S \]

**Solution:**
\[ (f \circ f^{-1})(S) = (ID)(S) = S \subseteq S \]

Problem 5:
How many people must be in a room to guarantee that at least 4 people have the same last initial.  **Answer is 79 people**

Problem 6:
If \( f: (\mathbb{Z}_{mod5}) \rightarrow (\mathbb{Z}_{mod5}) \) when \( f(x) = [3x + 1] \), determine if \( f \) is a bijection.

**Solution:**
\[
\begin{align*}
f[0] &= [1] \\
f[3] &= [0] \\
\end{align*}
\]