2534  Solutions for Equivalence Relation worksheet:

1) a) R1 is Reflexive and Symmetric  
b) R2 is Transitive  
c) R3 is Symmetric  
d) R4 is Symmetric  

2) $R = \{(1,1),(4,4),(1,4),(4,1),(3,3),(5,5),(3,5),(5,3),(2,2)\}$

3) Theorem: $a \equiv b \mod d$ is an equivalence relation on the integers: 
   ie. $a \ R \ b$ if and only if $d \mid (a-b)$
   Proof:
   R is reflexive if $aRa$ for all integers $a$. If $aRa$, then $d \mid a-a$. By definition of divisible there must exist an integer $q$ such that $dq = a-a$, Let $q = 0$ and the definition is valid. Therefore R is reflexive.
   
   R is symmetric if for all integers $a$, $b$, if $aRb$, then $bRa$. If it true that $aRb$ then $d \mid a-b$ and by definition of divisible $dp = a-b$ for some integer $p$. Multiply both sides by -1 to get $d(-p) = b-a$. So $dm = b-a$ where $m = -p$ is an integer. By definition of divisible we have that $d \mid b-a$ so $bRa$ and R is symmetric.
   
   R is transitive if for all integers $a$, $b$, $c$, if $aRb$ and $bRc$, then $aRc$. If it is true that $aRb$ and $bRc$ then by definition of divisible we have that $dk = a-b$ and $dh = b-c$. So adding $dk + dh = a-b + b-c$, we get $d(k+h) = a-b + b-c$. Therefore $dm = a-c$, where $m = k+h$ is an integer and $d \mid a-c$ by definition of divisible and $aRc$.
   
   Therefore we have shown that the relation R is an equivalence relation.
   
   The partition is $Z = [0] \cup [1] \cup [2] \cup [3] \cup \ldots \cup [d-1]$ 

4) Theorem: If relations R and S are each transitive, then $R \cap S$ is also transitive.
   Proof: If $(x,y)$ and $(y,z)$ are elements of $R \cap S$ then $(x,y)$ and $(y,z)$ are elements of $R$ and $(x,y)$ and $(y,z)$ are elements of $S$. Since R and S are each transitive, $(x, z)$ is in R and $(x, z)$ is in S which means that $(x, z)$ is in the intersection.