Math 2534 Solutions:
Worksheet on Equivalence Relations and Partial Order

1) A) Prove that \( a \equiv b \mod d \) is an equivalence relation on the integers.

By equivalent form of congruence we have that \( aRb \iff d \mid (a - b) \)

Proof: Reflexive: \( aRa \) since we can show that \( d \mid (a - a) \).

By definition of divisible we must have an integer \( q \), so that \( dq = a - a \).
If we let \( q = 0 \) we have satisfied the definition.

Symmetric: If \( aRb \), then we must show that \( bRa \). We are given that \( d \mid (a - b) \),
so there exist a \( k \) so that \( dk = a - b \). If we multiply both sides by \(-1\), we will get \( d(-k) = b - a \).
If we let \( p = -k \), we have that \( p \) is an integer and \( d \mid (b - a) \) by definition of divisible. Therefore \( bRa \) and \( R \) is symmetric.

Transitive: If \( aRb \) and \( bRc \), we must show that \( aRc \). We are given that \( d \mid (a - b) \) and \( d \mid (b - c) \), so there exist integers \( n \) and \( m \) so that \( dn = a - b \) and \( dm = b - c \) by definition of divisible. By adding \( dn + dm = a - b + b - c \), we have \( d(n + m) = a - c \). Let \( r = n + m \) so that \( dr = a - c \) and \( r \) is an integer.
Therefore \( d \mid (a - c) \) by definition of divisible and \( R \) is transitive.

B) Give the partition of integer induced by this congruency.
(Hint: Look at modular arithmetic)

2) If \( R \) and \( S \) are both transitive then prove that the intersection is also transitive.

Proof:
If \( (x, y), (y, z) \in R \cap S \), then \( (x, y), (y, z) \in R \) and \( (x, y), (y, z) \in S \)
Since \( R \) and \( S \) are both transitive, \( (x, z) \) is in \( R \) and \( (x, z) \) is in \( S \).
Therefore \( (x, z) \) must be in the intersection \( R \cap S \). So by definition of transitive, we have that \( R \cap S \) is also transitive.
3) Given a set A, for any a, b in A, aRb if and only if a divides b. Verify that R is a partial order on A.

Proof: Reflexive: We need to show that aRa, we need to show there exists an integer q so that $a|a$, or by definition of divisible $aq = a$. We can let $q = 1$ to satisfy the definition.

Ant-symmetric: If it is given that aRb and bRa, then we need to show that a = b. We are given that $a|b$ and $b|a$, so by definition of divisible there must be integers k and d so that $ak = b$ and $bd = a$. This will give us $(ak)d = a$, so that $a(kd) = a$. Since k and d are integers, we have that $kd = 1$ which gives that $k = 1$ and $d = 1$. Therefore $a = b$ and R is ant-symmetric.

Transitive: If it is given that aRb and bRc, then we must show that aRc. By definition of divisible we have integers q and p so that $aq = b$ and $bp = c$. This will give us $(aq)p = c$, which becomes $a(qp) = c$. Let $m = qp$ (which is an integer) to get $am = c$. So by definition of divisible we have that $a|c$ and aRc. Therefore R is transitive.

4) If $A = \{2, 3, 4, 6, 8, 9, 12, 18\}$ For all a, b in A, aRb iff a divides b.

Draw a Hasse Diagram representing R.

Solution: I can not easily come up with a electronic graph here, so I will give you the Relation in ordered pairs.

$R = \{(2, 2), (3, 3), (4, 4), (6, 6), (8, 8), (9, 9), (12, 12), (18, 18), (2, 4)(2, 6)\}

(2, 8), (2, 12), (2, 18), (3, 6), (3, 9), (3, 12), (3, 18), (4, 8), (4, 12), (6, 12)

(6, 18), (9, 18)\}$