Problem 1: If \( A = \{a, b, c\} \) and \( B = \{0, 1, 2, 3, 4, 5\} \) then define \( F : P(A) \to B \) as follows: For all subsets \( s \in P(A) \), \( f(s) = n(s) \) (ie. the number of elements in \( s \))

a) Show the precise mapping from the domain to the co-domain
b) Is \( f \) one to one? Why or why not?
c) Is \( f \) onto? Why or why not?

Solution:
\[
P(A) = \{\emptyset, \{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\} \}
\]
\[
f(\emptyset) = 0, \quad f(\{a, b, c\}) = 3, \quad f(\{a, b\}) = f(\{a, c\}) = f(\{b, c\}) = 2, \quad f(\{a\}) = f(\{b\}) = f(\{c\}) = 1
\]
\( f : P(A) \to B \) is not one to one or onto.

Problem 2:
If \( f : \mathbb{Z} \mod 4 \to \mathbb{Z} \mod 4 \) when \( f[x] = [4x+1] \),
Show the mapping and determine if \( f \) is a bijection.
\[
F[0] = [1], \quad F[1] = [5] = [1], \quad F[2] = [9] = [1], \quad F[3] = [13] = [1]
\]
This mapping is a constant mapping that is not one to one or onto.

Problem 3: If the functions \( f \) and \( g \) are both injections then the composition \((g \circ f)\) is also an injection. For the proof assume \( f : A \to B \) and \( g : B \to C \)

Solution: It given that \( f \) and \( g \) are each injections we need to prove that
If \( (g \circ f)(a_1) = (g \circ f)(a_2) \), Then \( a_1 = a_2 \)

Suppose that \( (g \circ f)(a_1) = (g \circ f)(a_2) \) and by definition of composite we have that
\( g(f(a_1)) = g(f(a_2)) \) and since \( g \) is an injection, we have that \( f(a_1) = f(a_2) \). Similarly, since \( f \) is an injection we also have that \( a_1 = a_2 \). So by definition of injection we have that the composite function is also an injection.

Problem 4: Prove that if \( F^{-1} \) exist and is a function, then \( F \) is one to one.
Proof: Assume that \( F : A \to B \) and \( F^{-1} \) exist. Assume that \( F \) is not one to one.
That means that there exist elements \( a_1 \) and \( a_2 \) so that \( F(a_1) = F(a_2) = b \). By definition of inverse relation we now have that \( F^{-1}(b) = a_1 \) and \( F^{-1}(b) = a_2 \). Therefore \( F^{-1} \) is not well defined and not a function which is a contradiction since \( F^{-1} \) is given to be a function.
Problem 5:
Let $C$ and $D$ be subsets of $A$ so that $A = C \cup D$ and $f : C \rightarrow B$ and $g : D \rightarrow B$. Define a function $h(x)$ as follows: $C$ and $D$ are disjoint sets.

$$h(x) = \begin{cases} f(x) & \text{if } x \in C \\ g(x) & \text{if } x \in D \end{cases}$$

Determine if the following is true or false and justify your conclusions.

a) If $h(x)$ is onto then $f(x)$ and $g(x)$ are each onto.
   This statement is false. Consider the following counter example:
   Let $C = \{1, 2\}$ and $D = \{3, 4\}$ and $B = \{a,b,c,d\}$
   $G(f) = \{(1,a), (2,b)\}$ and $G(g) = \{(3,c), (4,d)\}$ Notice that $h(x)$ is onto $B$ but neither $f$ or $g$ are onto.

b) If $f$ and $g$ is each one to one then $h$ is one to one.
   This statement is false. Consider the following counter example:
   Let $C = \{1, 2\}$ and $D = \{3, 4\}$ and $B = \{a,b\}$
   $G(f) = \{(1,a), (2,b)\}$ and $G(g) = \{(3,a), (4,b)\}$
   Notice that $f$ and $g$ are each one to one but $h(x)$ is not.

Problem 6: Given sets $A$ and $B$ and $f:A \rightarrow B$, if $n(A) > n(B)$ then $f(x)$ is not one to one.
Proof: Assume that $f$ is one to one. Then by definition of one to one if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$. So $n(A) = n(f(A))$. We also know that $f(A) \subseteq B$ which also means that $n(f(A)) \leq n(B)$. We now have that $n(A) = n(f(A)) \leq n(B)$ but this is a contradiction to the premises that $n(A) > n(B)$. Therefore $f$ is not one to one.

Problem 7: Given sets $A$ and $B$, If $A \subseteq B$, then $f(A) \subseteq f(B)$
Proof:
$$\forall y, y \in f(A) \rightarrow x \in A \quad \text{so that } f(x) = y$$
$$\rightarrow x \in B \quad \text{since } A \subseteq B$$
$$\rightarrow y \in f(B) \quad \text{so that } f(x) = y$$
Therefore $f(A) \subseteq f(B)$