Math 2534  Solution to Homework 6  on Proofs  Summer 2014

Problem 1:  Direct proof: (Using previous theorems only)
Theorem:  For all natural numbers, if \( a > 2 \), is a prime number, \( b \) is odd and \( c \) is even, then \( b(a^2 + cb + a) \) is even.

Proof:  It is given that \( a \) and \( b \) are prime numbers. Therefore we know that \( a \) and \( b \) are each odd. We know that \( a^2 = (a)(a) \) is also odd since the product of two odd numbers will always be odd. Since \( c \) is given to be even the product of \( cb \) is even because the product of an even integer and an odd integer is always even. We know that \( a^2 + cb \) will be odd since the sum of an odd and even integer is always odd. We also know that \( (a^2 + cb) + a \) is even, since the sum of two odd integers will always be even. Finally we have that \( b(a^2 + cb + a) \) is even, since the product of an odd and even integer is always even.

Problem 2:  Direct Proof: (Use definition of divisible)
Theorem:   If \( a \), \( b \) and \( c \) are natural numbers and \( c \) is divisible by \( d \) then \( ab \) is divisible by \( d \)

Proof:  It is given that \( ab \) is divisible by \( c \) and \( c \) is divisible by \( d \) so by definition of divisible we have that there exist integers \( k \) and \( q \) so that \( c = abk \) and \( d = cq \). This will give us that \( d = (abk)q = ab(kq) = (ab)m \) where \( m = kq \) is an integer. So by definition of divisible, we have that \( ab \) divides \( d \).

Problem 3:   Indirect Proof by contrapositive (contraposition) (use definitions only)
Theorem: Given \( a \) and \( b \) are integers, if the product \( ab \) is even then \( a \) is even or \( b \) is even.

Proof by contrapositive:   Restatement of the theorem will be as follows:
If \( a \) and \( b \) are odd then \( ab \) is odd.

Since we are given that \( a \) and \( b \) are each odd we have by definition of odd that there exist integers \( k \) and \( h \) so that \( a = 2k + 1 \) and \( b = 2h + 1 \).
Now consider the product \( ab = (2k + 1)(2h + 1) = 4kh + 2k + 2h + 1 = 2(kh + k + h) + 1 \)
= \( 2m + 1 \) where \( m = kh + k + h \) is also an integer. By the definition of odd, the product \( ab \) is odd. Since we have proved the contrapositive true, the equivalent original statement is also true and if the product \( ab \) is even then \( a \) is even or \( b \) is even.
Problem 4: Indirect Proof by contradiction (use definitions only)

Theorem: For all non-zero rational numbers, the product of a rational number and an irrational number is always irrational.

Proof by contradiction: Restatement will be as follows:
Assume that there exist a rational number r and an irrational number w and the product (r)(w) is rational.

Assuming that r is rational and the product (r)(w) is rational then by definition of rational we have that
\[ r = \frac{a}{b} \quad \text{and} \quad r \cdot w = \frac{c}{d} \]
where a, b, c, d are non-zero integers. Now consider the product
\[ r \cdot w = \frac{a}{b} \cdot \frac{c}{d} \]
and solve for w to get that
\[ w = \frac{c}{d} \cdot \frac{b}{a} = \frac{cb}{ad} = m \]
where
\[ m = cb \in \mathbb{Z}, \quad n = da \neq 0 \]
is in \( Z \) by definition of rational we have that w is also rational, but this is a contradiction to the premises that w is irrational. Therefore the product of a rational and an irrational number is irrational.

Problem 5: Proof is in two parts. One part is direct proof. For the other part use contrapositive approach.

Theorem: For any natural number n, n is even if and only if 7n + 4 is even.

Proof:
Part I: If the natural number n is even then 7n + 4 is even:
Proof: Given that n is even, we have by definition of even that there exist an integer k so that n = 2k. Now consider
\[ 7n + 4 = 7(2k) + 4 = 2[7k + 2] = 2p \]
where p = 7k + 2 is an integer. By definition of even, 7n + 4 is even.

Part II: If 7n + 4 is even then n is even.
Proof by contrapositive: Restatement is If n is odd, then 7n + 4 is odd.
Given that n is odd, by definition of odd there exist an integer f so that n = 2f + 1. Now consider
\[ 7(2f + 1) + 4 = 14f + 11 = 14f + 10 + 1 = 2(7f + 5) + 1 = 2p + 1 \]
where p = 7f + 5 is an integer. So by the definition of odd, 7n + 4 is odd. Since we have proved the contrapositive true, the equivalent original statement is also true. So if 7n + 4 is even then n is even.

Therefore by the proof of part I and part II we have proved that n is even if and only if 7n + 4 is even.