Math 2534 Solution to Homework 10 Fall 2013

Problem 1:
A) Prove that \( a \equiv b \mod 6 \) is an equivalence relation \( R \) on the integers.

This means that \( aRb \) if and on if \( 6|(a-b) \)

Proof: To prove that the relation \( R \) is an equivalence on the integers we must prove that \( R \) is reflexive, symmetric, and transitive.

Reflexive: To show that \( aRa \), we must show that \( 6|(a-a) \). We must show that there exist an integer \( q \) such that \( 6q = a-a \). In order for \( 6q = 0 \), \( q = 0 \) and this will satisfy the definition of divisible and \( aRa \). And \( R \) is reflexive.

Symmetric: Given that \( aRb \), we need to show that \( bRa \). By definition of divisible we have that \( 6k = a-b \). By multiplying both sides by \((-1)\) we will have \( 6(-k) = b-a \). Let \( p = -k \) where \( p \) is an integer, then by definition of divisible we have that \( bRa \) and \( R \) is symmetric.

Transitive: Given that \( aRb \) and \( bRc \), we need to show that \( aRc \). By definition of divisible we have that \( 6q = a-b \) and \( 6p = b-c \).

Now consider \( 6q + 6p = a-b + b-c = a-c \). Let \( k = q + p \) to get \( 6k = a-c \).

By definition of divisible we have that \( 6 \) divides \( a-c \) and \( aRc \). Therefore \( R \) is transitive.

B) Give the partition of integers induced by this congruency.


Problem 2:
If \( R \) and \( S \) are both symmetric then prove that the intersection is also symmetric.

Proof:
We are given that \( S \) and \( R \) are each symmetric.

Consider the point \((a, b)\) in \( S \cap R \). Then by definition of intersection we know that \((a,b)\) is in \( S \) and \((a,b)\) is in \( R \). Since \( S \) is symmetric we know that \((b,a)\) is also in \( S \) and similarly we know that \((b,a)\) is in \( R \). Therefore by definition of intersection we know that \((a,b)\) must be in \( S \cap R \). Therefore \( S \cap R \) is also symmetric.

Problem 3:
Set \( A = \{a, b, c, d, e, f\} \) is partitioned into \( A = A = \{a\} \cup \{c, e, f\} \cup \{d, b\} \).

Give the equivalence \( R \) that produces this partition.

\[ R = \{(a,a),(c,c),(e,e),(f,f),(e,c),(c,e),(c,f),(f,c),(e,f),(f,e),(d,d),(b,b),(d,b),(d,b)\} \]
Problem 4:
Determine if the following statements are correct. If not, then explain why not.
a) \( f : X \rightarrow Y \) is onto if and only if \( \forall x \in X, \exists y \in Y \) so that \( f(x) = y \).
**FALSE:** This is just the definition of a function.

b) \( f : X \rightarrow Y \) is one to one if and only if \( \forall x \in X, \exists y \in Y \) so that \( f(x) = y \).
**FALSE:** This is just the definition of a function.

c) \( \) Let \( f : X \rightarrow Y \). A sufficient condition for \( f(x) \) to be one to one is that \( \forall y \in Y, \) there is at most one \( x \in X \), with \( f(x) = y \).
**TRUE:**
d) \( f : X \rightarrow Y \) is onto if and only if the range and the co-domain are the same.
**TRUE:**

Problem 5: Let the function \( h(x) \) map set \( A \) to set \( B \). Let \( C \) and \( D \) be disjoint subsets so that \( C \cup D = A \). Define functions \( f : C \rightarrow B \) and \( g : D \rightarrow B \) so that \( h(x) = f(x) \) for all \( x \) in \( C \) and \( h(x) = g(x) \) for all \( x \) in \( D \). If \( f(x) \) is one to one and \( g(x) \) is onto, is \( h(x) \) a bijection? Justify your conclusion. (Hint: look at examples)
**FALSE:** Let \( C = \{ 1, 2 \} \), \( D = \{ 3, 4, 5, 6 \} \), \( B = \{ a, b, c \} \) Consider the following example.
\( f : C \rightarrow B \) so that \( f(1) = a \) and \( f(2) = b \)
\( g : D \rightarrow B \) so that \( g(3) = a \), \( g(4) = b \), \( g(5) = b \) and \( g(6) = c \)
Notice that \( f \) is one to one and \( g \) is onto but \( h(x) \) cannot be one to one and is not a bijection.

Problem 6: Explain the mistake in the following proof:
Theorem: If \( f(x) = 4x + 3 \) for all integers, then \( f(x) \) is one to one.

**Proof:** Suppose any integer \( x \) is given. Then by definition of \( f \), there is only one possible value for \( f(x) \), namely \( 4x + 3 \). Hence \( f(x) \) is one to one.

**Solution:** This only confirms that \( f(x) \) is a function. It does not consider the concept of one to one.

Problem 7:
How many students have to be in the same class to guaranteed that 8 students in that class have the same last initial? You would need 183 people.

Problem 8:
State the **precise formal definition** for two sets \( A \) and \( B \) to have the same cardinality.
Two sets have the same cardinality if and only if there exist a bijection that maps one set to the other.