Math 2534  Solution  Homework 9

Problem 1: Use elements to prove the following theorems:

Theorem 1: If $A \subseteq B$, then $A - C \subseteq B - C$ for all sets $A$, $B$, $C$.

Proof:
\[
\forall x, x \in A - C \rightarrow x \in A \land x \notin C \quad \text{by definition of Difference} \\
\rightarrow x \in B \land x \notin C \quad \text{since } A \subseteq B \\
\rightarrow x \in (B - C) \quad \text{by definition of Difference} \\
\therefore A - C \subseteq B - C
\]

Theorem 2: $P(A) \cup P(B) \subseteq P(A \cup B)$ for all sets $A$ and $B$.

Proof:
\[
\forall x, x \in P(A) \cup P(B) \rightarrow x \in P(A) \lor x \in P(B) \quad \text{by definition of union} \\
\rightarrow x \subseteq A \lor x \subseteq B \quad \text{by definition of power set} \\
\rightarrow x \subseteq (A \cup B) \quad \text{by definition of union} \\
\rightarrow x \in P(A \cup B) \quad \text{by definition of Power set} \\
\therefore P(A) \cup P(B) \subseteq P(A \cup B)
\]

Problem 2: Use Algebra of sets to prove the following theorems given the sets $A$ and $B$.

Theorem 4: The symmetric difference $A \oplus B = (A \cup B) - (A \cap B)$ may also be expressed as $A \oplus B = (A - B) \cup (B - A)$. (Hint: Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$)

Proof:
\[
(A \cup B) - (A \cap B) \equiv \quad \text{given} \\
(A \cup B) \cap (A \cap B)^c \equiv \quad \text{By definition of Difference} \\
(A \cup B) \cap (A^c \cup B^c) \equiv \quad \text{DeMorgan's Law} \\
[(A \cup B) \cap A^c] \cup [(A \cap B) \cap B^c] \equiv \quad \text{By Distribution Law} \\
[(A \cap A^c) \cup (B \cap A^c)] \cup [(A \cap B^c) \cup (B \cap B^c)] \equiv \quad \text{By Distribution Law} \\
[\emptyset \cup (B \cap A^c)] \cup [(A \cap B^c) \cup \emptyset] \equiv \quad \text{By Complement Law} \\
(B \cap A^c) \cup (A \cap B^c) \equiv \quad \text{By Identity Law} \\
(B - A) \cup (A - B) \equiv \quad \text{By definition of Difference} \\
(A - B) \cup (B - A) \equiv \quad \text{By Commutative Law} \\
\therefore (A \cup B) - (A \cap B) \equiv (A - B) \cup (B - A)
\]
Problem 3: Prove using set Algebra prove the following theorems given sets A and B.

Theorem: \((A^C \cup B^C) - A\)^C = A

Proof

\[\begin{align*}
(A^C \cup B^C) - A\}^C &\equiv \text{ Given} \\
[(A^C \cup B^C) \cap A^C]^C &\equiv \text{ Equivalent form of Difference} \\
[A^C]^C &\equiv \text{ By Absorption} \\
A &\equiv \text{ Double Complement}
\end{align*}\]

Problem 4: Using the formula \( n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \) solve the following.

Five hundred V.T students were surveyed to find out how many made use of the Emporium. In particular tutoring, study areas and computers. Two hundred and eighty five use the tutoring, one hundred and ninety five use the study areas, one hundred and fifteen use the computers, forty five use both tutoring and computers, seventy use tutoring and study areas, fifty use the study areas and computers and fifty used none of these.

a) How many used all three?
b) How many used study areas only?
c) How many used computers only?

Answers:

a) 15 students
b) 90 students
c) 40 students

Problem 5:

Given the Boolean Algebra \(B\) with operations \(\odot\) and \(\ominus\) where \(h\) is the identity for \(\odot\) and \(k\) is the identity for \(\ominus\), prove the following theorem by supplying the justification for each step.

Theorem: For all elements \(a, b\) in \(B\), \((a \ominus b) \ominus a = a\)

Proof:

\[\begin{align*}
(a \ominus b) \ominus a &\equiv \text{ Given} \\
(a \odot a) \odot (b \ominus a) &\equiv \text{ distribution} \\
a \odot (b \ominus a) &\equiv \text{ Idempotent} \\
a \odot (a \ominus b) &\equiv \text{ commutative}
\end{align*}\]
\[(a \otimes k) \otimes (a \otimes b) = \text{Identity } k \text{ for } \otimes\]
\[a \otimes (k \otimes b) = \text{distribution}\]
\[a \otimes k = \text{Dominance Law}\]
\[a = \text{Identity for } \otimes\]