Math 2214 Solutions Test 2A Spring 2010

Problem 1: Solve the initial value problem below:

\[ y'' + 25y = 0, \quad y(0) = -3, \quad y'(0) = 2 \]

Solution: (12pts)

\[ y'' + 25y = 0, \quad y(0) = -3, \quad y'(0) = 2 \]

\[ m^2 + 25m + 0 \quad \text{and} \quad m = \pm 5i \]

\[ y = C_1 \cos(5t) + C_2 \sin(5t) \quad \text{where} \quad C_1 = -3 \quad \text{and} \quad C_2 = 5/2 \]

Problem 2:

Find the general solution for 

\[ y'' - y' - 2y = 4e^{2t} \quad \text{(always consider the possibility that } g(t) \text{ is a solution for the homogeneous form)} \]

Solution: (15pts)

\[ y'' - y' - 2y = 4e^{2t} \]

\[ y_H = C_1 e^{2t} + C_2 e^{-t} \]

Choose \( y_p = A e^{2t} \) but if \( C_2 = 0, \) and \( C_1 = A, \) this is a solution for the homogeneous form. Instead choose \( y_p = Ate^{2t} \) and \( A = 4/3 \)

\[ \therefore y = C_1 e^{2t} + C_2 e^{-t} + (4/3)te^{2t} \]

Problem 3:

Given the equation 

\[ y''' + 4y' = t^3 e^t \sin(4t); \]

a) find the general solution for the homogeneous equation

b) Give the set up for the particular solution, but do not solve for the unknown constants.

Solution: (10pts)

\[ y = y_H + y_p \]

\[ = C_1 e^{9t} + C_2 \cos 2t + C_3 \sin 2t + (A_1 t^3 + A_2 t^2 + A_3 t + A_4) e^t \cos 4t + (B_1 t^3 + B_2 t^2 + B_3 t + B_4) e^t \sin 4t \]
Solution: (4pts)

Problem 4: Suppose that we know that \( y = e^{-t} \sin(6t) \) is a solution for differential equation \( ay'' + by' + cy = 0 \), where \( a, b, \) and \( c \) are constants. Which of the following must also be solutions? Check all that apply.

___X_0 (zero),    _____1 (one),        __X___ 3e^{-t} \cos(6t),    ___X__ e^{-t}(2 \cos(6t) - 5 \sin(6t))

_____e^{3t}(\cos(6t) + \sin(6t)),    _____e^{-t}(\cos(3t) + \sin(3t))),    _____e^{-t} \sin(12t)

____X__5 e^{-t} \sin(6t),    _____e^t \sin(6t)

Problem 5: (18pts)

Use the method of variation of parameters to find the general solution for the following problem, \( t^2 \frac{d^2 y}{dt^2} - 4t \frac{dy}{dt} + 6y = t^3 + 2t^2, y > 0 \). Given \( y_1 = t^2 \) and \( y_2 = t^3 \) are solutions to the homogeneous form, first verify that \( y_1 \) and \( y_2 \) form a fundamental set.

Solution:

\( t^2 \frac{d^2 y}{dt^2} - 4t \frac{dy}{dt} + 6y = t^3 + 2t^2, y > 0 \)

\( y'' - \frac{4}{t} y' + \frac{6}{t^2} y = t + 2 \) for \( t \neq 0 \)

\[ W = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = t^4 \neq 0 \text{ for } t \neq 0 \]

\[ y_H = C_1 t^2 + C_2 t^3 \]

Choose \( y_p = U_1 t^2 + U_2 t^3 \)

Set up the needed system to be

\[ \begin{cases} U_1 t^2 + U_2 t^3 = 0 \\ 2U_1 t + U_2 t^3 = t + 2 \end{cases} \]

To get

\[ \begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix} \begin{bmatrix} U_1' \\ U_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t + 2 \end{bmatrix} \]

\[ \begin{bmatrix} U_1' \\ U_2' \end{bmatrix} = \begin{bmatrix} 3t^2 & -t^3 \\ -2t & t^2 \end{bmatrix} \begin{bmatrix} 0 \\ t + 2 \end{bmatrix} = \begin{bmatrix} -1 - 2/t \\ 1/2 - 2/t^2 \end{bmatrix} \]

\[ U_1 = \int U_1' dt = -t - 2 \ln t + D \text{ (let } D = 0) \]

\[ U_2 = \int U_2' dt = t - 2 \ln t + K \text{ (let } K = 0) \]

Solution is \( y = C_1 t^2 + C_2 t^3 + (-t - 2 \ln t)t^2 + (\ln t - 2/t)t^3 \)
**Problem 6: (12pts)**

Suppose you are given the solutions $y_1 = e^{-t}$ and $y_2$ where $y_2$ is unknown, but $y_2(0) = 2$ for the differential equation $y'' + ay' + by = 0$, $a$ and $b$ are constants. Using the Wronskian find the solution $y_2$ when the Wronskian is equal to $4e^{2t}$.

**Solution:**

$$\begin{vmatrix} e^{-t} & y_2' \\ -e^{-t} & y_2' \end{vmatrix} = e^{-t}y_2' + e^{-t}y_2 = 4e^{2t}$$

So $y_2' + y_2 = 4e^{3t}$ let the integrating factor $u = e^t$

$$\int [y_2 e^t] dt = \int 4e^{4t} dt$$

$y_2 e^t = e^{4t} + k$

$y_2 = e^{3t} + ke^{-t}$ where $k = 1$ since $y_2(0) = 2$

**Problem 7:** Initial value problem $u'' + 6u' + 14u = 12\cos(2t)$ models the spring mass system and has solution $u = \frac{31e^{-3t}}{61}\cos(\sqrt{5}t) + \frac{143e^{-3t}}{61\sqrt{5}}\sin(\sqrt{5}t) + \frac{30}{61}\cos(2t) + \frac{36}{61}\sin(2t)$

a) Would this solution be a transient solution or resonance solution? Justify your conclusion by loosely sketching a quick graph.

(4pts) Transient

The graph will start out erratic and then damp to zero and then take on a simple harmonic motion.

b) If this system was unforced, then It would best be described as undamped(simple harmonic), Underdamped, overdamped or critically damped.

(3pts) Underdamped

c) Considering the particular solution only (not the homogeneous solution), determine the amplitude and period. (Set up only and do not do the actual calculations)

(8pts) $R = \sqrt{(30/61)^2 + (36/61)^2}$

Period = $2\pi/2 = \pi$
Problem 8: An object weighing 8 lbs is attached to a steel spring of length 15 in. The weight of the object extends the spring a distance of 1/8 ft to equilibrium rest. The object is also attached to an oil dashpot damper with a damping constant of $c$.

Set up a complete initial value equation that models this spring motion and determine the value of $c$ for which the motion would be critically damped.

(15pts)

\[
\frac{1}{4}s'' + cs' + 64s = 0
\]

The system is critically damped when the characteristic equation has repeated roots.

\[
\frac{1}{4}m^2 + cm + 64 = 0
\]

\[
m^2 + 4cm + 256 = 0
\]

so $m = \frac{-4c \pm \sqrt{16c^2 - 4(256)}}{2}$ and repeated roots occur when $16c^2 - 4(256) = 0$

so $C = 8$ and \[
\frac{1}{4}s'' + 8s' + 64s = 0
\]