Problem 1: Use separation of variables to solve the Initial value DE \( y' = te^{t-y}, \ y(0) = 0 \)

Solution (14pts)

\[ y' = te^{t-y}, \ y(0) = 0 \]

\[ \frac{dy}{dt} = te^t e^{-y} \]

\[ \int e^y dy = \int te^t dt \] using u-du by letting \( u = t^2 \)

\[ e^y = \frac{e^{t^2}}{2} + \text{ where } C = \frac{1}{2} \] for the initial value.

so \( y = \ln \left| \frac{e^{t^2} + 1}{2} \right| \)

Problem 2: Given that \( y = e^{-3t} + t - 3 \) is a unique solution for the initial value DE \( y' + p(t)y = g(t) \) and \( y(0) = y_0 \). Find \( p(t) \), \( g(t) \) and \( y_0 \).

Solution (10pts) consider the solution \( y \):

\( y = e^{-3t} + t - 3 \) first we have \( y_0 = e^{-3(0)} + 0 - 3 = 1 - 3 = -2 \)

Now divide by \( e^{-3t} \) to get that

\( ye^{3t} = 1 + te^{3t} - 3e^{3t} \) now consider the derivative of both sides

\[ [ye^{3t}]' = 0 + 3te^{3t} + e^{3t} - 9e^{3t} = e^{3t}[3t + 1 - 9] = e^{3t}[3t - 8] \]

From the process of solving the first order linear DE we know that the integrating factor is \( u = e^{3t} \), \( e^{\int 3dt} \) so \( p(t) = 3 \)

and \( g(t) = 3t - 8 \)
**Problem 3:** Given the initial value problem \( y' - \frac{t}{y^2} = 0, \quad y(-4) = 2 \), use Euler’s Method to evaluate \( y(-2) \) when step value is \( h = 1 \).

**Solution (10pts)**

\( y' - \frac{t}{y^2} = 0, \quad y(-4) = 2 \)

Using the point \((-4, 2)\) and the slope \( m = y' = -1 \)
the equation of the tangent line is \( y = -(t+2) \) and \( y(-3) = 1 \)

Next using the new point \((-3,1)\) and slope \( m = y' = -3 \)
the equation of the tangent line is \( y = -3t - 8 \) and \( y(-2) = -2 \)

**Problem 4:** What is the largest \( t \)-interval for which \( (t-4)y' - \frac{y}{t+2} = \frac{\sec t}{t} \) has a guaranteed unique solution for initial value \( y(-1) = 3 \) according to the Existence–Uniqueness Theorem?

**Solution (8pts)**

\( (t-4)y' - \frac{y}{t+2} = \frac{\sec t}{t} \)

\( y' = \frac{y}{(t-4)(t+2)} = \frac{\sec t}{t(t+2)} \)

\( t \neq 4, t \neq 0, t \neq -2, t \neq (n\pi)/2 \) and \( y(-1) = 3 \)

The largest interval for this initial value is \((-\pi/2, 0)\)
Problem 5: An object weighing 19.6 Newtons is dropped downward with an initial velocity of 20 m/sec. The magnitude of the force on the object due to air resistance is \( R = \frac{S}{20} \).

a) Is resistance a positive or negative force in this scenario?
b) Is velocity a positive or negative force in this scenario?

b) Set up the differential equation that models this scenario in terms of speed. (Do not solve)
c) Set up the differential equation again in terms of velocity and give initial value. (Do not solve)

Solution (12 pts)

\( a) \) Resistance is positive
\( b) \) Velocity is negative

\( c) \)

\[
mv' = -mg + \frac{s}{20}
\]

\[
2v' = -19.6 + \frac{s}{20}
\]

\[
v' = -9.8 + \frac{s}{20}
\]

\( d) \)

\[
2v' = -19.6 - \frac{v}{20} \quad \text{since} \quad s = |v| = -v \quad \text{since velocity is negative}
\]

\[
v' = -9.8 - \frac{v}{40}
\]

Problem 6: Consider the initial value non-linear DE \( y' = \frac{\sqrt{y}}{t^2 - 1} \) with \( y(0) = 0 \)

Is a unique solution guaranteed for this initial value? Justify your conclusion.

\( y' = \frac{\sqrt{y}}{(t^2 - 1)} \) with \( y(0) = 0 \) \hspace{1cm} \text{In order to determine the open region that would guarantee}

a unique solution for this initial value we will determine where the following is continuous.

1) \( f(t, y) = y' = \frac{\sqrt{y}}{(t^2 - 1)} \) where \( t \neq 1, t \neq -1, t \geq 0 \)

2) \( \frac{\partial f}{\partial y} = \frac{1}{2(t^2 - 1)\sqrt{y}} \) where \( t \neq 1, t \neq -1, t \neq 0, t > 0 \)

However the partial derivative is not defined for the initial value so the solution is not guaranteed.
**Problem 7:** Below is an example of the logistic equation which describes growth with a natural population ceiling: Determine the differential equation below that is represented by this direction field.

**Explain your reasoning for your choice. Explain why you rejected the other possibilities.**

![Direction Field](image)

a) \( y' = y(y - 15) \)

b) \( y' = y(15 - y) \)

c) neither a) or b)

The equilibrium solutions occur at \( y = 0 \) and \( y = 15 \) for both a) and b).

Now test other isoclines:

For a) let \( y = 20 \) to get slopes \( y' = 20(20 - 15) = 100 \) which is positive but the actual slope is negative. Now test b) at \( y = 20 \) to get \( y' = -100 \) which is the correct sign. Now we need to check b) at other isoclines to be sure that it really is the correct DE. Test \( y = 10 \) to get \( y' = 50 \) Which is positive and matches. Now test \( y = -10 \) to get \( y' = -250 \) and has the correct sign. Therefore b) is the correct.
Problem 8: Solve the following first order linear DE using the integrating factor.

\[ t^2 y' + 2ty = t + 1, \quad y(1) = -1 \]

Solution (14 pts)

\[ t^2 y' + 2ty = t + 1, \quad y(1) = -1 \]

\[ y' + \frac{2}{t} y = \frac{t+1}{t^2} \quad \text{and} \quad \text{integrating factor is} \ u(t) = e^{\int \frac{2}{t} dt} = e^{\ln(t^2)} = t^2 \]

so \[ \int [y t^2] \, dt = \int \frac{t+1}{t^2} \, t^2 \, dt \]

\[ y t^2 = \frac{t^2}{2} + t + C \]

\[ y = \frac{1}{2} + t^{-1} + Ct^{-2} \quad \text{using initial value} \quad C = \frac{-5}{2} \quad \text{and} \quad y = \frac{1}{2} + \frac{1}{t} - \frac{5}{2t^2} \]

Problem 9: A tank has a capacity for 800 gallons and contains 300 gallons of water with 7 lbs of salt initially. A solution containing 6 lbs/gal of salt is pumped into the tank at 10 gals/min. A well stirred mixture flows out of the tank at the same rate.

a) Set up the differential equation that models this situation.
b) Solve by using the integrating factor to find the equation for the amount of salt in the tank at any time t. (show all work in detail)
c) Find and explain the limiting value of your solution.
d) Suppose the rate at which the solution flows out changes to 8 gals/min. Set up the differential equation that models this new situation. (Do not solve this DE.)
e) Determine how long it will take to fill the tank using the situation in part d).