Problem 1:

a) \( y' + \frac{1}{t} y = \sin t \), \( y(\pi) = 4 \)  
The largest interval that will guarantee a solution is \((0, \infty)\)

solution: let \( u(t) = e^{\int \frac{1}{t} \, dt} = e^{\ln t} = t \)

Multiply diff. eq. by \( u(t) \) to get

\[
y'(t) + y(t) = t \sin t \quad \text{and due to the product rule to get} \quad [y(t)]' = t \sin t
\]

\[
\int [y(t)]' \, dt = \int t \sin t \, dt \quad \text{(using integration by parts)}
\]

\[
y(t) = -t \cos t - \int -\cos t \, dt = -t \cos t + \sin t + C
\]

using the initial value \( y(\pi) = 4 \), \( 4 = 1 + 0 + C \) / \( \pi \) so \( C = 3\pi \)

so \( y = -t \cos t + \sin t / t + (3\pi) / t \), The \( \lim_{t \to \infty} y(t) \) oscillates between 1, -1

b) \((t+1)y' + y = 6\), \( y(1) = -2 \)  
(largest Interval for initial value \((-1, \infty)\))

First rewrite diff. eq to be \( y' + \frac{1}{t+1} y = \frac{6}{t+1} \)

using \( u(t) = e^{\int \frac{1}{t+1} \, dt} = e^{\ln(t+1)} = t+1 \) we have \([y(t+1)]' = 6 \)

\[
\int [y(t+1)]' \, dt = \int 6 \, dt
\]

\[
y(t+1) = 6t + C, \text{ using initial value we find } C = -10 \text{ so } y = \frac{6t-10}{t+1}
\]

\( \lim_{t \to \infty} y = 6 \)

(c) \( y' + \sin(t)y = 0 \), \( y(\pi / 2) = 3 \)  
The largest interval for initial value is \((-\infty, \infty)\)

multiply diff eq with \( u(t) = e^{\int \sin(t) \, dt} = e^{-\cos t} \) to get \([y e^{-\cos t}]' = 0 \)

\[
\int [y e^{-\cos t}]' \, dt = \int 0 \, dt \quad \text{integrate to get } y e^{-\cos t} = C \quad \text{so the solution } y = C e^{\cos t}
\]

using the initial value \( 3 = C e^{\cos(\pi / 2)} \) so \( C = 3 \) and \( y = 3e^{\cos t} \)

\( \lim_{t \to \infty} y = \text{oscillates between } 3e \text{ and } 3/e \)
Problem 2:
Initially a tank contains 40 pounds of salt dissolved in 100 gallons of fresh water. A solution containing 1/4 lb of salt per gallon is added to the tank at a rate of 3 gal/min. At the same time a “well stirred” solution is leaving the tank at the same rate. Find the equation for the amount of salt in the tank at any time t. Determine the limiting factor for this solution.

Solution:
\[ Q'(t) = (0.25)(3) - \frac{3Q(t)}{100} \]
\[ Q'(t) - \frac{3}{100}Q = 0.75 \quad \text{using } u(t) = e^{\int 0.03 \, dt} = e^{0.03t} \quad \text{we have that} \]
\[ \int [Qe^{0.03t}] \, dt = \int 0.75e^{0.03t} \, dt \]
\[ Qe^{0.03t} = \frac{0.75e^{0.03t}}{0.03} + C \]
\[ Q = 25 + Ce^{-0.03t} \quad \text{using the initial value } Q(0) = 40 \quad \text{we have } C = 15 \]
So \[ Q = 25 + 15e^{-0.03t} \quad \text{and Limit } Q = 25 \text{lbs of salt} \]

Extra Problem for notes:
Redo Problem 2 with the following change. The “well stirred” solution is leaving the tank at the rate of 4 gallon per minute.

\[ Q'(t) = (0.25)(3) - \frac{2Q(t)}{100-t} \]
\[ Q'(t) - \frac{2}{100-t}Q = 0.75 \quad \text{using } u(t) = e^{\int \frac{2}{100-t} \, dt} = e^{-2\ln(100-t)} = (100-t)^{-2} \quad \text{we have that} \]
\[ \int [Q(100-t)^{-2}] \, dt = \int 0.75(100-t)^{-2} \, dt \]
\[ Q(100-t)^{-2} = -0.25(100-t)^{-1} + C \]
\[ Q = -0.25(100-t) + C(100-t)^2 \quad \text{use the initial value } Q(0) = 40 \quad \text{to find } C \]