Problem 1: Suppose a viral disease is spreading through a large population at a rate of change modeled by the differential equation \( Q' = kQ^2 \) where the initial number of people infected was equal to 100. In a week a total of 500 people were infected. Use the method of separation of variables to solve for the number people infected at any time \( t \).

Solution: (16pts)

\[
Q'(0) = 100
\]

\[
\int \frac{dQ}{Q^2} = kdt
\]

\[
\frac{-1}{Q} = kt + C \quad \text{and with initial values this gives us} \quad C = \frac{-1}{100}
\]

\[
\frac{-1}{Q} = kt + \frac{1}{100} \quad \text{and with value} \quad t = 1\text{week and} \quad Q = 500, k = \frac{4}{500}
\]

so

\[
Q = \frac{1}{\frac{1}{100} - \frac{t}{125}}
\]

Problem 2: Use Euler’s method (not Euler’s formula) to evaluate \( y(3) \) when you are given the following non-linear differential equation:

\[
y' = t^2 - y, \quad \text{with initial value} \quad y(1) = 2 \quad \text{and step size is} \quad h = 1.
\]

Discuss if you think the estimate would be close. Explain why or why not.

Solution: (16pts)

the method involves finding a point, slope and the equation of the tangent line. Then evaluate the point of interest.

Part 1) We are given the point (1, 2) and the slope is \( y' = (1)^2 - 2 = -1 \) and the equation is \( y - 2 = (-1)(t - 1) \) to simplify to \( y = -t + 3 \) and \( y(2) = 1 \)

Part 2) We now have the point (2, 1) and the slope is \( y' = (2)^2 - 1 = 3 \) and the equation is \( y - 1 = (3)(t - 2) \) to simplify to \( y = 3t - 5 \) and \( y(3) = 4 \)

In general the step value between the \( t \) values is probably too large to be a close estimation.
Problem 3: A body of weight 128 lbs is dropped with an initial velocity of 2 ft/sec. The resistance constant due to air is given to be $k = 8$. Using the given differential equation, find the equation for velocity at any time $t$. 

$$mv' = -mg - kv$$

Solution: (16pts)
Since the weight is given to be 128 lbs we know that $mg = m(32) = 128$ so $m = 4$. Using $mv' = -mg - kv$ and inserting appropriate values we have

$$4v' = -128 - 8v$$
$$v' + 2v = -32$$

find the integrating factor to be $u = e^{2t}$

so 

$$\int [ve^{2t}]' dt = \int -32e^{2t} dt$$
$$ve^{2t} = -16e^{2t} + C$$
$$v = -16 + Ce^{-2t}$$

using $v(0) = -2$ (since velocity is negative)

$$-2 = -16 + C \quad \text{and} \quad C = 14, \quad \text{so} \quad v = -16 + 14e^{-2t}$$

Problem 4: A 400 gallon open top tank initially holds 100 gallons of fresh water. At $t = 0$, a brine solution containing 2 lb. of salt per gallon is poured into the tank at a rate of 3 gallons per minute while a well stirred solution leaves the tank at a rate of 1 gallon per minute.

Solution: (8pts)

a) Set up the differential equation that models this scenario. (do not solve)

$$Q' + \frac{Q}{100 + 2t} = 6, \quad Q(0) = 0$$

Solution: (4pts)

c) At what time will overflow occur?

Volume $= 100 + 2t = 400$, so $t = 150$ min.

Problem 5: Using Newton’s law of cooling we have the temperature of a given object at any time $t$ is given to be $T(t) = 80 - 30e^{-2t}$ °F

Solution: (8pts)

a) What was the initial temperature of the object?

At $t = 0$, $T(0) = 80 - 30e^0 = 80 - 30 = 50$

b) What was the limiting temperature for the object?

$$\text{Limit}_{t \to \infty} (80 - 30e^{-2t}) = 80 - 0 = 80$$

I would like to comment here that $\infty$ is not a number value and you should not try to use this notation in your calculations. Also, when the limit is zero then you should write the zero in the appropriate place.
Problem 6: Demonstrate how to use the Existence and Uniqueness Theorem to find the largest rectangular region that guarantees a unique solution for the non-linear differential equation 
\[ y'(t-2) = \tan(y), \text{ with initial value } y(\pi) = 1/2. \] Sketch the region.

Solution: (12pts)

\[ f(t, y) = y' = \frac{\tan y}{t-2} \]
\[ \frac{\partial f}{\partial y} = \frac{\sec^2 y}{t-2} \]

*notice* 2 < t < ∞ and −π/2 < y < π/2 for the initial value y(π) = 1/2

To sketch the region you need to plot the initial value point and enclose it in the rectangular region that is bounded by the intervals given above.

Problem 7: Given the differential equation
\[ y' + p(t)y = g(t) \]
with the particular solution
\[ y = 2 + e^{-5t^2}, \text{ find } p(t) \text{ and } g(t). \] Show all work and explain reasoning.

Solutions: (10pts)

Consider the given solution:
\[ y = 2 + e^{-5t^2} \]
\[ ye^{5t^2} = 2e^{5t^2} + 1 \text{ Take the derivative on both sides to get the following:} \]
\[ [ye^{5t^2}]' = [2e^{5t^2} + 1]' \]
\[ y' e^{5t^2} + \left(10te^{5t^2}\right) y = 20te^{5t^2} + 0 \]
\[ [y' + 10ty = 20t]e^{5t^2} \]

Clearly the integrating factor is \( u(t) = e^{5t^2}, p(t) = 10t \) and \( g(t) = 20t \)
Problem 8: Given the following direction field, find the possible autonomous differential equation associated with it and clearly justify your answer.

\[ y' = (y - 3)(y + 1) \]

\[ y' = -(y - 3)(y + 1) \]

\[ y' = -(y - 3)(y + 1) \]

Solution (10 pts)
We first notice the equilibrium isoclines are found at \( y = -1 \) and \( y = -3 \).
By investigating the equations above we notice that when we set \( y' = 0 \) we have the following results:

1) \( 0 = (y - 3)(y + 1) \) and so \( y = 3 \) and \( y = -1 \) which matches up with the equilibrium isocline \( y = -1 \) but not \( y = -3 \).

2) \( 0 = - (y - 3)(y + 1) \) and so \( y = 3 \) and \( y = -1 \) which matches up with the equilibrium isocline \( y = -1 \) but not \( y = -3 \).

3) \( 0 = - (y - 3)(y + 1) \) and so \( y = -3 \) and \( y = -1 \) which matches up with both the equilibrium isoclines \( y = -1 \) and \( y = -3 \).

Since 3) is the only possibility in these choices, we will check to see if it will actually produce the direction field. We will test the slope when \( y = 0 \). So \( y' = ( - 0 - 3)(0+ 1) = -3 \) which a negative slope that corresponds with the graph. Now check \( y = -2 \), \( y' = (2 - 3)(-2+ 1) = 1 \) which also corresponds with the graph. Therefore \( y' = (-y - 3)(y + 1) \) produces this direction field.