Sample Final Problems since Second Test

1. Let $X \in \mathfrak{sl}(2, \mathbb{C})$. Prove that $X^2 = -\det(X)I$.

2. Let $G$ be an abelian Lie group. Prove that $G$ is connected if and only if the exponential map $\exp : \mathfrak{g} \to G$ is onto.

Solutions

1. We know there exists $A \in \text{GL}(2, \mathbb{C})$ such that $AXA^{-1}$ is in Jordan canonical form. Since $\text{tr}(AXA^{-1}) = \text{tr}(X) = 0$, $AX^2A^{-1} = (AXA^{-1})^2$ and $\det(XAX^{-1}) = \det(X)$, we may assume that $A$ is in Jordan canonical form, and that the eigenvalues $a, b$ satisfy $a + b = 0$. If $a \neq b$, then $A = \text{diag}(a, -a)$ and the result is clear. On the other hand if $a = 0$, then either $A = 0$ or $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the result follows because in this case $A^2 = 0$ and $\det(A) = 0$.

2. Since $\exp$ is continuous and $\mathfrak{g}$ is connected, it follows that the image of $\exp$ is also connected. This proves the “only if” part. Conversely suppose $G$ is connected. We know that $\exp(\mathfrak{g})$ generates $G$ as a group. However when $G$ is abelian, let $X, Y \in \mathfrak{g}$. Intuitively it is obvious that $[X, Y] = 0$, but I think this needs proof. For $g \in G$, let $\gamma_g : G \to G$ denote conjugation by $g$, so $\gamma_g(h) = ghg^{-1}$, and let $A \in G$. Then $\exp(\text{Ad}_A(tX)) = \gamma_A(\exp(tX)) = \exp(tX)$ for all $t \in \mathbb{R}$. Since $\exp$ is one-to-one in a neighborhood of 0, we deduce that $\text{Ad}_A = I$, the identity. Furthermore $\text{Ad}_{\exp(\text{ad}X)} = \exp(t \text{ad}X)$, and we similarly deduce that $\text{ad}_X = 0$. But we also know that $\text{ad}_X(Y) = [X, Y]$, thus $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Therefore $\exp(X + Y) = \exp(X)\exp(Y)$, i.e. $\exp$ is a group homomorphism. Since $\exp(\mathfrak{g})$ generates $G$ as a group, it follows that $\exp(\mathfrak{g}) = G$ as required.

Exam on Monday May 13 in the regular classroom McBryde 318. There will be 5 problems, one of them will be identical to (part of) one of the homework problems (graded or ungraded) or one of the sample final problems since the second test. The other four problems will come from material for the two tests.