Fifth Homework Solutions

2. Exercise 3.6 on page 45.

Let \( G \) be the group of complex matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \(|a| = 1\), with the usual topology.

(a) Is this group compact?
(b) Show that the fundamental representation of \( G \) on \( \mathbb{C}^2 \) is reducible, but not completely reducible.
(c) Find the endomorphisms of \( \mathbb{C}^2 \) that commute with the fundamental representation of \( G \).

(a) The group is not compact because it is not bounded, in particular \(|b|\) can be arbitrarily large.
(b) For convenience, we write column vectors as row vectors. The fundamental representation is reducible because \( \mathbb{C}(1,0) \) is a one-dimensional invariant subspace. We note that \( \mathbb{C}(1,0) \) is the only one-dimensional invariant subspace. Indeed if \( \mathbb{C}v \) is a one-dimensional invariant subspace, then in particular \( v \) is an eigenvector of \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). However \( v \) is the only eigenvector of this matrix. Thus \( \mathbb{C}v \) cannot have an invariant direct complement.
(c) We need to find the matrices which commute with all elements of \( G \). Let \( e_{pq} \) denote the matrix units, so \( e_{pq} \) has 1 in the \((p,q)\)th position and zeros elsewhere. Suppose \( \alpha := \sum_{i,j} a_{ij} e_{ij} \) commutes with all elements of \( G \), where \( a_{ij} \in \mathbb{C} \) for all \( i, j \). Then in particular, \( \alpha e_{pq} = e_{pq} \alpha \) for all \( p \leq q \). Using \( e_{pq} e_{ij} = e_{pi} \delta_{qi} \), we find that

\[
   a_{1p} e_{1q} + a_{2p} e_{2q} = a_{1q} e_{p1} + a_{2q} e_{p2}.
\]

Then \( p = q = 1 \) yields \( a_{12} = a_{21} = 0 \), and \( p = 1, q = 2 \) yields \( a_{11} = a_{22} \). This shows that the matrices which commute with all elements of \( G \) are precisely scalar multiples of the identity matrix.

4. Exercise 3.7(b) on page 45.

Find the left and right invariant measures on the group of affine transformations of \( \mathbb{R} \).

The affine group \( G \) consists of matrices

\[
   \{m(x,y) := \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x,y \in \mathbb{R}, x \neq 0 \}.
\]

We’ll identify \( G \) via \( m \) with \( \{(x,y) \mid x,y \in \mathbb{R}, x \neq 0 \} \).
First we find left Haar measure. We need to find a measure which is invariant under the map $(x, y) \mapsto m(p, q)(x, y) = (px, py + q)$. We claim that $\mu(R) := \iint_R \frac{dydx}{x^2}$ will do, for every measurable subset $R$ of $G$. It will be sufficient to check this on rectangular regions $(a, b) \times (c, d)$. Note that the map sends rectangular regions to rectangular regions, so we need to prove that

$$\int_a^c \int_b^d \frac{dydx}{x^2} = \int_{pa}^{pc} \int_{pb+q}^{qd+q} \frac{dydx}{x^2}.$$

This is true because both sides are $(1/a - 1/c)(d - b)$.

Now we prove right invariance. We claim that $\mu(R) := \iint_R \frac{dydx}{|x|}$ will do, for every measurable subset of $G$ (we need the $|x|$, because $x$ can take negative values, and by definition of Haar measure, it cannot take negative values). It will be sufficient to check this on rectangular regions $(a, b) \times (c, d)$. By breaking up the region into parts where $x$ is positive and $x$ is negative, we can reduce to the case $a, c > 0$. Now the general element of $G$ is the product of elements of the form $(p, 0)$ and $(1, q)$, so it will be sufficient to show that $\mu$ is invariant under these two types of elements. First we consider the map $(x, y) \mapsto (x, y)m(p, 0) = (px, y)$. Note that this map sends rectangles to rectangles, so we need to prove

$$\int_a^c \int_b^d \frac{dydx}{x} = \int_{pa}^{pc} \int_{pb}^{pd+q} \frac{dydx}{x}.$$

This is true because both sides are $(d - b)\ln(c/a)$. Now we consider the map $(x, y) \mapsto (x, y)m(1, q) = (x, qx + y)$. Note that this map sends rectangles to parallelograms, so more care is required. We need to prove

$$\int_a^c \int_b^d \frac{dydx}{x} = \int_{b+qx}^{b+q} \int_{b+q}^{d+qy} \frac{dydx}{x}.$$

This is again true because both sides are $(d - b)\ln(c/a)$. 