Tenth Homework Solutions

1. Let \( A \in \mathfrak{sl}(2, \mathbb{R}) \). If \( \{a, b\} \) are the eigenvalues of \( A \), then we have \( a + b = 0 \) and \( ab \in \mathbb{R} \). Thus \( a = -b \) and \( a^2 \in \mathbb{R} \). Therefore \( a \) is either real or pure imaginary. It follows that the eigenvalues of \( A \) are \( \{\lambda, -\lambda\} \) or \( \{i\lambda, -i\lambda\} \), where \( \lambda \in \mathbb{R} \). Note that if \( A \) is similar to \( B \), then \( \exp(A) \) is similar to \( \exp(B) \), because \( \exp(XBX^{-1}) = X \exp(B)X^{-1} \) for an invertible matrix \( X \), and it follows that \( \exp(A) \) and \( \exp(B) \) have the same trace. If the eigenvalues are not distinct, then the only possibility for \( \lambda \) is 0, and this case \( A \) is similar to the zero matrix or \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), and in both cases it is clear that \( \text{tr}(\exp(A)) \geq 0 \).

If the eigenvalues of \( A \) are distinct, then \( A \) is similar to a diagonal matrix with the eigenvalues on the main diagonal. Since \( \exp(i\lambda) \geq -1 \), we see that \( \text{tr}(\exp(A)) \geq -2 \) for all \( A \in \mathfrak{sl}(2, \mathbb{R}) \). Clearly there are matrices in \( \text{SL}(2, \mathbb{R}) \) such that this is not the case.

Unfortunately the question is about \( \mathfrak{sl}(2, \mathbb{C}) \), not \( \mathfrak{sl}(2, \mathbb{R}) \). Here is a sketch proof for this. I claim that the nondiagonalizable matrix \( A := \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \) is not in the image of the exponential map. If \( X \in \mathfrak{sl}(2, \mathbb{C}) \) is diagonalizable, then so is \( \exp(X) \). On the other hand if \( X \) is not diagonalizable, then since the trace of \( X \) is 0, it is similar to the matrix \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). But then \( \exp(X) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), which is not similar to \( A \).

The exponential map is not always injective. For example take \( G = U(1) \). Then \( \mathfrak{g} = \mathbb{R} \), and the exponential map is \( t \mapsto e^{it} \).

2. Let \( Y \in \mathfrak{h} \). Then for all \( X \in \mathfrak{g} \),

\[
[X, Y] \in \mathfrak{h} \iff \exp(X)Y\exp(-X) \in \mathfrak{h} \quad \text{by ungraded homework of April 19}
\]

\[
\iff \text{Ad}_{\exp X} Y \in \mathfrak{h}
\]

\[
\iff \exp(X)\exp(Y)\exp(-X) \in \mathcal{H}
\]

\[
\iff g\exp(Y)g^{-1} \in \mathcal{H} \quad \text{for all } g \in G, \text{because } G \text{ is connected.}
\]

Since this is true for all \( X \in \mathfrak{g} \), it now follows that \( \mathfrak{h} \) is an ideal of \( \mathfrak{g} \) if and only if \( \mathcal{H} \triangleleft G \), because \( \mathcal{H} \) is connected.