First Test. Answer All Problems.
Please Give Explanations For Your Answers

1. Prove that $Q_8 \times S_3$ is not isomorphic to $D_{48}$. ($Q_8$ denotes the quaternion group of order 8 and $D_{48}$ denotes the dihedral group of order 48.) (12 points)

2. Let $G$ be a group. Prove that the formula $(g, h) \cdot x = gxh^{-1}$ for $g, h, x \in G$, defines an action of $G \times G$ on $G$. Show further that if $Z(G) \neq 1$, then there exists a nonidentity element which acts trivially on $G$ (i.e. there exists $1 \neq k \in G \times G$ such that $k \cdot x = x$ for all $x \in G$). (12 points)

3. Prove that if $A$ and $B$ are subsets of the group $G$ with $A \subseteq B$, then $C_G(B) \leq C_G(A)$. Is it always true that $N_G(B) \leq N_G(A)$? Justify your answer. ($C_G$ and $N_G$ denote centralizer and normalizer) (12 points)

4. Let $D_{16} = \langle r, s \mid r^8 = 1, s^2 = 1, rs = sr^{-1} \rangle$, the dihedral group of order 16. Determine for which positive integers $n$ there is a subgroup of order $n$ in $D_{16}$ (so you will need to give an example of a subgroup of order $n$ for such integers, and explain why there cannot be subgroups of other orders). (12 points)

5. Prove that if $G/Z(G)$ is cyclic, then $G$ is abelian ($Z(G)$ denotes the center of $G$). (12 points)
Solutions to First Test

1. The elements of $Q_8$ have orders 1, 2 and 4, while the orders of the elements of $S_3$ have orders 1, 2 and 3. Since $|(x,y)| = |[x], |y]|$, we see that the largest possible order of an element of $Q_8 \times S_3$ is $4 \times 3 = 12$. However $D_{48}$ has an element of order 24.

2. To show that we have an action, we have two things to check (note that $(1,1)$ is the identity of $G \times G$).

   (i) $(1,1) \cdot x = 1x1^{-1} = x.$

   (ii) $(p,q) \cdot ((g,h) \cdot x) = (p,q) \cdot (gxy^{-1}) = pgxy^{-1} = (pg)x(qh)^{-1}$ (remember that the inverse of a product is the product of the inverses in the reverse order) $= ((p,q)(g,h)) \cdot x.$

   Finally if $Z(G) \neq 1$, there exists $z \in Z(G) \setminus 1$. Now set $k = (z,z)$, which is not equal to 1. Then $k \cdot x = (z,z) \cdot x = xzz^{-1} = xzz^{-1} = z.$

3. Recall that $C_G(A) = \{ g \in G \mid ga = ag \text{ for all } a \in A \}$. First note that $1 \in C_G(A)$, because $1a = a = a1$ for all $a \in A$. Next if $g, h \in C_G(A)$, then $ga = ag$ and $ha = ah$ for all $a \in A$, hence $gha = gah = ag = gh$ for all $a \in A$ and we see that $gh \in C_G(A)$. Finally if $g \in C_G(A)$, then $ga = ag$ for all $a \in A$, hence $g^{-1}ag^{-1} = g^{-1}ag^{-1}$ and we see that $g^{-1}a = ag^{-1}$ for all $a \in A$ and we deduce that $g^{-1} \in C_G(A)$. Therefore $C_G(A) \subseteq G$. Similarly $C_G(G) \subseteq G$.

   We now need to prove containment. Let $x \in C_G(B)$. This means that $x$ commutes with all elements of $B$. Since $A \subseteq B$, we immediately see that $x$ commutes with all elements of $A$ and we conclude that $x \in C_G(A)$ as required.

   Finally we take $B = G = S_3$ and $A = \{(1,2)\}$. Then $A \subseteq B$ and $N_G(B) = S_3$. On the other hand $\{(1,3)\} \not\subseteq N_G(A)$, because $(1,3)(1,2)(1,3)^{-1} = (2,3)$. Thus $N_G(B) \not\subseteq N_G(A)$, so the answer is no.

4. By Lagrange’s theorem, the order of a subgroup divides 16, so the only possibilities are 1,2,4,8,16. Now we certainly have 16, by using $G$. Also $r$ has order 8, so $\langle r \rangle$ is a cyclic subgroup of order 8. Since a cyclic subgroup has a subgroup of order for all numbers dividing its order, 8, it follows that $G$ also has subgroups of order 1, 2, 4 and 8 (namely $\langle r^8 \rangle$, $\langle r^4 \rangle$, $\langle r^2 \rangle$, $\langle r \rangle$). Thus the answer is 1,2,4,8,16.

5. Write $Z = Z(G)$. If $G/Z$ is cyclic, then $G/Z = \langle xZ \rangle$ for some $x \in G$. This means that the left cosets of $Z$ in $G$ are of the form $x^aZ$ for some $a \in Z$ and $z \in Z$. Therefore every element of $G$ is of the form $x^a z$. Suppose we have another such element, say $x^b w$ (so $w \in Z$). Then

   $$(x^az)(x^bw) = zwx^a x^b = wzx^b x^a = (wx^b)(zx^a),$$

   consequently any two elements of $G$ commute and we have proven that $G$ is abelian.