Solutions to Sample Second Test

1. We have $HN/N \cong H/H \cap N$. Since we are dealing with finite groups, by taking orders we obtain $|HN|/|N| = |H|/(H \cap N)$, hence $|HN|/|H| = |N|/(N \cap H)$ and we deduce that $|HN/H| = |N/N \cap H|$. But $|HN/H|$ divides $|G/H|$ and $|N/N \cap H|$ divides $|N|$. Since $(|G/H|, |N|) = 1$, we deduce that $|N/N \cap H| = 1$. Therefore $N = N \cap H$ which shows that $N \subseteq H$.

2. If $g \in G$, then $tg \neq g$. This means when we write out $\pi t$ as a product of disjoint cycles, it cannot contain any 1-cycles; also $t^2 g = 1g = g$, so the other cycles are all 2-cycles. In other words $\pi t$ is a product of 12 disjoint 2-cycles (each of the cycles will be of the form $(g, tg)$ for some $g \in G$). In particular $\pi t \in A_G \cong A_{24}$.

If $g \in S_4$, then $g$ is a product of transpositions (not disjoint in general, and certainly not uniquely so) so we may write $g = t_1 \ldots t_n$ where the $t_i$ are transpositions. Then $\pi g = (\pi t_1) \ldots (\pi t_n)$ because $\pi$ is a homomorphism. From the first paragraph, $\pi t_i \in A_G$ for all $i$ and we deduce that $\pi g \in A_G$.

Finally $\ker \pi = 1$, so $\pi$ is a one-to-one homomorphism whose image is contained in $A_G$. We deduce that $G$ is isomorphic to a subgroup of $A_{24}$.

3. First we compute the size of the conjugacy class. We have $10 \cdot 9 \cdot \ldots \cdot 3$ ways to choose the eight entries. Then we can rotate each cycle; that means we have to divide by $3 \cdot 3 \cdot 2$. Finally we can interchange the two 3-cycles; that means we need to divide by 2. Therefore the total number is $\frac{10 \cdot 9 \ldots 3}{3 \cdot 3 \cdot 2 \cdot 2}$. Thus the size of the conjugacy class containing $(1 2 3)(4 5 6)(7 8)$ is 50400. Also the order of the centralizer this element is $10!/50400 = 72$.

4. Write $Z = Z(G)$. Since $G$ is nonabelian, $Z \neq G$. Also the center of a $p$-group is not 1 (unless the group is 1). It now follows from Lagrange’s theorem that $|Z| = p, p^2$ or $p^3$; we need to rule out $|Z| = p^3$. But if this is the case, then $|G/Z| = p$ and since groups of prime order are cyclic, we deduce that $G$ is abelian. This is a contradiction, so the result is proven.

5. Write $D_{12}$ in the usual way, that is $\{r,s \mid r^6 = s^2 = 1, rs = sr^{-1}\}$. Then $D_{12}$ is generated by $r, s$, so we can bound the order of $D_{12}$ by considering the possibilities for $\theta r$ and $\theta s$ where $\theta \in \text{Aut}(D_{12})$. Now $r$ has order 6, so $\theta r$ also has order 6; there are only two elements of order 6 in $D_{12}$, namely $r$ and $r^{-1}$. Therefore there are two possibilities for $\theta r$. Also $s$ has order 2, so $\theta s$ also has order 2. There are seven elements of order 2 in $D_{12}$, namely $s, sr, sr^2, sr^3, sr^4, sr^5$ and $r^3$. It is not difficult to rule out the case $\theta s = r^3$, but we will proceed by a different method. Since $\theta$ is determined by its values on a generating set, we now have at most $2 \cdot 7$ possibilities for $\theta$, in other words $|\text{Aut}(D_{12})| \leq 14$.

Now $Z(D_{12}) = \{1, r^3\}$, so $|D_{12}/Z(D_{12})| = 6$. Since $\text{Inn}(D_{12}) \cong D_{12}/Z(D_{12})$, we deduce that $|\text{Inn}(D_{12})| = 6$. But $\text{Inn}(D_{12}) \leq \text{Aut}(D_{12})$, so by Lagrange’s theorem we deduce that 6 divides $|\text{Aut}(D_{12})|$ and the result follows.
6. Let $G$ be a group of order $5^2 \cdot 7^2 \cdot 17$. This question is a bit nasty because normally you would consider the Sylow 17-subgroups first, but this doesn’t lead anywhere immediately because there is the possibility of 35 Sylow 17-subgroups. And then the next try would normally be the Sylow 7-subgroups, but this also doesn’t help because there could be 85 Sylow 7-subgroups. In fact here one considers the Sylow 5-subgroups first. The number of Sylow 5-subgroups is congruent to 1 modulo 5 and divides $7^2 \cdot 17$; the only possibility is 1 and therefore $G$ has a normal Sylow 5-subgroup $A$ say.

Now consider $G/A$. This is a group of order $|G|/|A| = 7^2 \cdot 17$. The number of Sylow 17-subgroups is congruent to 1 modulo 17 and divides 49, hence there is only one Sylow 17-subgroup, which by the subgroup correspondence theorem we may call $H/A$ for some subgroup $H$ of $G$ containing $A$, and since $H/A \triangleleft G/A$ we have $H \triangleleft G$. Also $|H| = |H/A||A| = 17 \cdot 5^2$. The number of Sylow 17-subgroups in $H$ is congruent to 1 modulo 17 and divides $7^2$; the only possibility is 1, and we see that $H$ has exactly one Sylow 17-subgroup, which we shall call $B$. We now show that $B$ is not only normal in $H$, but is also normal in $G$. Let $g \in G$. Then $gBg^{-1} \leq gHg^{-1} = H$ (because $H \triangleleft G$) and $|gBg^{-1}| = |B|$, hence $gBg^{-1}$ is a Sylow 17-subgroup of $H$. Since $H$ has only one Sylow 17-subgroup, we must have $gBg^{-1} = B$ and this establishes that $B \triangleleft G$.

Using Sylow’s theorem on $G/A$, we see that the number of Sylow 7-subgroups is congruent to 1 modulo 7 and divides 17, hence $G/A$ has a unique Sylow 7-subgroup which by the subgroup correspondence theorem we may call $K/A$ for some subgroup $K$ of $G$ containing $A$. Then $K \triangleleft G$ because $K/A \triangleleft G/A$, and $|K| = |K/A||A| = 7^2 \cdot 5^2$. The number of Sylow 7-subgroups in $K$ is congruent to 1 modulo 7 and divides 25; the only possibility is 1, and we see that $K$ has exactly one Sylow 7-subgroup, which we shall call $C$. We now show that $C$ is not only normal in $K$, but is also normal in $G$. Let $g \in G$. Then $gCg^{-1} \leq gKg^{-1} = K$ (because $K \triangleleft G$) and $|gCg^{-1}| = |C|$, hence $gCg^{-1}$ is a Sylow 7-subgroup of $K$. Since $K$ has only one Sylow 7-subgroup, we must have $gCg^{-1} = C$ and this establishes that $C \triangleleft G$.

Thus $G$ has normal subgroups $A, B, C$ with orders 25, 17, 49 respectively. By Lagrange $A \cap B = A \cap C = 1$, consequently every element of $A$ commutes with every element of $B$ and $C$. Thus $B, C \leq C_G(A)$. Also $A$ has prime squared order, so is abelian and we deduce that $A \leq C_G(A)$. Therefore $A, B, C \leq C_G(A)$, so by Lagrange $5^2 \cdot 17 \cdot 7$ divides $|C_G(A)|$. We conclude that $C_G(A) = G$, consequently $A \leq Z(G)$. Similarly $B, C \leq Z(G)$. Again by Lagrange $|A|, |B|, |C|$ divide $|Z(G)|$, so we must have $|Z(G)| = G$ which means that $G$ is abelian, as required.

Test on Monday, April 4. Review session on Sunday, April 3 at 4:45 p.m. in McBryde 304. Material sections 3.3, 3.5, and chapter 4.

One of the problems will be identical to one of the ungraded homework problems since the first test (won’t ask Exercise 3.2.11 on page 96 from March 14, but you should certainly know the result that if $H \leq K \leq G$, then $|G/H| = |G/K| \cdot |K/H|$; this is easy if $G$ is finite).