Ninth Homework Solutions

1. Exercise 7.3.10. Decide which of the following are ideals of the ring $\mathbb{Z}[x]$.

Let $I$ be the relevant subset of $\mathbb{Z}[x]$.

(a) The set of all polynomials whose constant term is a multiple of 3.

Yes. Obviously $0 \in I$ and $I$ is an abelian group under addition. Finally suppose $f = a_0 + a_1 x + \cdots + a_m x^m \in I$ and $g = b_0 + b_1 x + \cdots + b_n x^n \in \mathbb{Z}[x]$. Then $3$ divides $a_0$ and $fg = a_0 b_0 + (a_1 b_0 + a_0 b_1) x + \cdots + a_m b_n x^{m+n}$. Since $3$ divides $a_0 b_0$, we see that $fg \in I$ and we have proven that $I$ is an ideal.

(b) The set of all polynomials whose coefficient of $x^2$ is a multiple of 3.

No. $x \in I$ yet $x^2 \notin I$, so $I$ is not even a subring.

(c) The set of all polynomials whose constant term, coefficient of $x$ and coefficient of $x^2$ are zero.

Yes. This is just the ideal $x^3 \mathbb{Z}[x]$.

(d) $\mathbb{Z}[x^2]$.

No. $1 \in I$ but $1x \notin I$ (on the other hand, $I$ is a subring).

2. Exercise 7.3.34. Let $I$ and $J$ be ideals of the ring $R$ with a 1.

(a) Prove that $I + J$ is the smallest ideal of $R$ containing both $I$ and $J$.

(b) Prove that $IJ$ is an ideal contained in $I \cap J$.

(c) Give an example where $IJ \neq I \cap J$.

(d) Prove that if $R$ is commutative and if $I + J = R$, then $IJ = I \cap J$.

(a) Obviously $I + J$ is an ideal containing both $I$ and $J$. Suppose $K$ is an ideal containing both $I$ and $J$. Since $K$ is closed under addition, it must contain $I + J$. Therefore $I + J$ is the smallest ideal containing $I$ and $J$.

(b) Since $I \triangleleft R$, we have $IJ \subseteq IR \subseteq I$. Similarly $IJ \subseteq J$ and it follows that $IJ \subseteq I \cap J$.

(c) Let $R = \mathbb{Z}$ and $I = J = 2\mathbb{Z}$. Then $IJ = 4\mathbb{Z}$ and $I \cap J = 2\mathbb{Z}$.

(d) In view of (b), we need to prove that if $x \in I \cap J$, then $x \in IJ$. Since $R = I + J$, we may write $1 = i + j$ where $i \in I$ and $j \in J$. Then $x = ix + xj \in IJ + IJ = IJ$ as required.

3. Define a ring homomorphism $\theta : \mathbb{R}[x] \to \mathbb{C}$ by $\theta(r) = r$ for $r \in \mathbb{R}$ and $\theta(x) = i$. Then $\theta$ is onto, because if $a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$, then $a + bi = \theta(a + bx)$. By the fundamental homomorphism theorem $\mathbb{R}[x]/\ker \theta \cong \mathbb{C}$. Set $M = \ker \theta$. Then $M \triangleleft \mathbb{R}[x]$ (because $\ker \theta$ is an ideal). Finally $M$ is a maximal ideal because $\mathbb{R}[x]/M$ is a field.
4. There are several ways to do this. One way is to use the isomorphism \((R/I)/(M/I) \cong R/M\). Then \(M/I\) is a maximal ideal of \(R/I\) tells us that \((R/I)/(M/I)\) is a field, hence \(R/M\) is a field and consequently \(M\) is a maximal ideal of \(R\).

5. Consider the ring \(R := \mathbb{Z}/6\mathbb{Z}\). Then \(P := 3\mathbb{Z}/6\mathbb{Z}\) is a maximal ideal of \(R\), because \(R/P \cong \mathbb{Z}/3\mathbb{Z}\) which is a field, so certainly \(P\) is a prime ideal of \(R\). It consists of two elements, namely 0 and \(\bar{3}\). Thus if \(P^2 \neq P\), then \(P^2\) has strictly less than two elements and hence \(P^2 = 0\). This in particular would tell us that \(\bar{3}^2 = 0\). But \(\bar{3}^2 = \bar{9} = \bar{3} \neq 0\) and we conclude that \(P^2 = P\).