Tenth Homework Solutions

1. (a) Let \( s, t \in S \) and suppose \( st \notin S \). Then \( stx = 0 \) for some nonzero \( x \in R \). Then \( tx \neq 0 \) because \( t \) is a nonzero divisor, and hence \( stx \) because \( s \) is a nonzero divisor and \( tx \neq 0 \). It follows that \( S \) is a multiplicatively closed subset of \( R \). Obviously it contains 1 but not 0.

(b) Let \( r/s \in S^{-1}R \), where \( r \in R \) and \( s \in S \). If \( r \) is a zero divisor, then \( rx = 0 \) for some nonzero \( x \in R \), hence \((r/s)(x/1) = 0 \) and \( x/1 \neq 0 \). This shows that \( r/s \) is a zero divisor. On the other hand if \( r \) is not a zero divisor, then \( r \in S \) by definition of \( S \) and thus \( s/r \in S^{-1}R \). Since \((r/s)(s/r) = 1 \), we see that \( r/s \) is a unit and we’re finished.

2. Let \( K = \ker \alpha \) and \( J = \ker \beta \). Then \( R/K \cong \mathbb{Z}/2\mathbb{Z} \), a field, and we see that \( K \) is a maximal ideal of \( R \). Similarly \( J \) is a maximal ideal of \( R \). Since \( |R/J| \neq |R/K| \), we see that \( J \neq K \) and hence \( J + K = R \). By the Chinese Remainder theorem, we see that

\[ R/(I \cap J) \cong R/I \times R/J \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Q}. \]

It follows that there is a ring epimorphism \( \theta : R \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Q} \).

3. We need to show that if \( p/1 \) divides \((a/s)(b/t)\) in \( S^{-1}R \), then \( p/1 \) divides either \((a/s)\) or \((b/t)\). Since divisibility is not affected by multiplying by a unit, we may assume that \( p/1 \) divides \( ab/1 \), so \((p/1)(r/x) = ab/1 \), for some \( r/x \in S^{-1}R \). Since \( R \) is an integral domain, this tells us that \( pr = abx \). Since \( p \) is prime, we see that \( p \) divides \( a \) or \( b \) or \( x \). If \( p \) divides \( x \), then \( p \) is a unit which contradicts the hypothesis that \( p \) is a prime, so without loss of generality we may assume that \( p \) divides \( a \). We conclude that \( p/1 \) divides \( a/1 \) and the result follows.

4. Exercise 9.4.7 on p. 311. Prove that \( \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C} \).

As in homework 9 problem 3, define \( \theta : \mathbb{R}[x] \rightarrow \mathbb{C} \) by \( \theta(r) = r \) for \( r \in \mathbb{R} \) and \( \theta(x) = i \). Then \( \theta \) is onto. Also \( \theta(x^2 + 1) = (\theta x)^2 + \theta(1) = 0 \), so \( x^2 + 1 \subseteq \ker \theta \). Suppose \( f \in \mathbb{R}[x] \) and \( \theta(f) = 0 \). By the division algorithm, we may write \( f = (x^2 + 1)q + r \) where \( q, r \in \mathbb{R}[x] \) and deg \( r \leq 1 \). Then \( \theta(r) = 0 \). Write \( r = ax + b \) where \( a, b \in \mathbb{R} \). Then \( \theta(r) = a + ib \); this can be 0 only when \( a = b = 0 \). This shows that \( f \in (x^2 + 1) \) and hence \( \ker \theta = (x^2 + 1) \). The result now follows from the fundamental homomorphism theorem.