**Notation:** Noting the formulas for \( c_0 \) and \( c_n, n \geq 1 \), it’s been found convenient to replace \( c_0 \) by \( c_0/2 \), i.e.,

\[
\begin{align*}
\omega(x,t) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(\frac{\pi nx}{L}) \\
\omega(x,0) &= f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(\frac{\pi nx}{L}),
\end{align*}
\]

because, since \( \cos(\pi) = 1 \), we’ll have the following single formula for \( \{c_n\}_{n=0}^{\infty} \):

\[
C_n = \frac{2}{T} \int_0^T f(x) \cos(\frac{\pi nx}{L}) dx, \quad n = 0, 1, 2, 3, \ldots
\]

Now, however:

\[
\frac{c_0}{2} = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x) dx
\]

**Example:** Solve:

\[
\begin{align*}
U_t(x,t) &= 0.1 \quad U_{xx}(x,t), \quad 0 < x < 3, \quad 0 < t < \infty \\
U_t(x,0) &= U_x(3,t) = 0, \quad 0 < x < \infty \\
U(x,0) &= f(x) = \begin{cases} 
3, & 1 \leq x \leq 2 \\
0, & 0 \leq x < 1 \text{ and } 2 < x \leq 3
\end{cases}
\end{align*}
\]

Adopting the above notation: \( \omega(x,t) = c_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{\pi nx}{L}) \)

\[
U(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} C_n \cos(\frac{\pi nx}{L}) = f(x), \quad 0 \leq x \leq 3.
\]

\[
C_n = \frac{2}{T} \int_0^T f(x) \cos(\frac{\pi nx}{L}) dx = \frac{2}{\pi} \int_1^3 \cos(\frac{\pi nx}{L}) dx = 2 \int_1^3 \sin(\frac{\pi nx}{L}) \frac{dx}{\pi} = 2 \int_1^3 \sin(\frac{\pi n}{3}) dx = 2 \int_1^3 \sin(\frac{\pi n}{3}) dx = 2 \int_1^3 \sin(\frac{\pi n}{3}) dx
\]

\[
C_n = \frac{2}{\pi} \left( \sin \left( \frac{2\pi n}{3} \right) - \sin \left( \frac{\pi n}{3} \right) \right), \quad n = 0, 1, 2, \ldots
\]

Note that this expression is indeterminate when \( n = 0 \). Either compute:

\[
\lim_{n \to 0} \frac{2}{\pi} \left( \sin \left( \frac{2\pi n}{3} \right) - \sin \left( \frac{\pi n}{3} \right) \right) = 2 \text{ or compute } C_0 = \frac{2}{\pi} \int_1^3 dx = 2.
\]

The solution is plotted on the next page, at times \( t = 0, 0.5 \), and \( 3 \), using a sum of the first 100 terms as an approximation to the solution.

Note that after the \( C_n \) coefficients are defined, \( C_0 \) is set equal to 2 to remove the indeterminacy before plotting.

At \( t = 0 \), the Gibbs Phenomena are evident at the discontinuities at \( x = 1 \) and \( x = 2 \). The smoothing effect of the heat equation is also evident in the plots of \( U \) at times \( t = 0.5 \) and \( 3 \).
Problem:
\[ u_t(x,t) = 0.1 u_{xx}(x,t), \quad 0 < x < 3, \quad 0 < t < \infty \]
\[ u_x(0,t) = u_x(3,t) = 0, \quad 0 < t \leq \infty \]
\[ u(x,0) = f(x) = \begin{cases} 
3, & 1 \leq x \leq 2 \\
0 & 0 \leq x < 1 \text{ and } 2 < x \leq 3 
\end{cases} \]

\[ 2 \cdot \text{Integrate}[\cos[n \pi + x / 3], \{x, 1, 2\}] \]
\[ 6 \left( -\sin\left[ \frac{n \pi}{3} \right] + \sin\left[ \frac{2n \pi}{3} \right] \right) \]

\[ \frac{n \pi}{n \pi} \]

Coefficients \( c[n] \) are determined.
\[ c[n_] := \frac{6 \left( -\sin\left[ \frac{n \pi}{3} \right] + \sin\left[ \frac{2n \pi}{3} \right] \right)}{n \pi}; \]

The indeterminacy at \( n = 0 \) is removed.
\[ \text{Limit}\left[ \frac{6 \left( -\sin\left[ \frac{n \pi}{3} \right] + \sin\left[ \frac{2n \pi}{3} \right] \right)}{n \pi}, \ n \to 0 \right] \]
\[ 2 \]
\[ c[0] = 2; \]

\[ u[x_\_, t_\_, N_\_] := \]
\[ c[0] / 2 + \text{Sum}[c[n] \cdot \exp[-((n \pi / 3)^2) \cdot 0.1 \cdot t] \cdot \cos[n \pi + x / 3], \{n, 1, N\}] \]

A 100 term sum approximating the initial condition is plotted. Note the presence of Gibbs Phenomena at the discontinuities

\[ \text{Plot}[u[x, 0, 100], \{x, 0, 3\}] \]

The solution at times \( t = 0, 0.5 \) and 3 are plotted. Note the smoothing effect of the heat equation upon the initial condition discontinuities.

As anticipated the solution profile tends toward the constant value 1 as time \( t \) increases.
Plot[{u[x, 0, 100], u[x, 0.5, 100], u[x, 3, 100]}, {x, 0, 3}]
Constant Temperature Ends:

We now consider the problem where the two ends of the bar are maintained at constant temperatures, say \( u(0,t) = T_0 \) and \( u(l,t) = T_1 \), \( 0 \leq t < \infty \).

**Problem:**

\[
\begin{align*}
&u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < l, \quad 0 < t < \infty \\
&u(0,t) = T_0, \quad u(l,t) = T_1, \quad 0 \leq t < \infty \\
&u(x,0) = f(x), \quad 0 \leq x \leq l
\end{align*}
\]

To be compatible with the boundary conditions we assume \( f(0) = T_0, \ f(l) = T_1 \).

**Note:** The boundary conditions are nonhomogeneous. Therefore, we must alter the approach previously used. If each "building block" solution, \( u_n \), satisfies the boundary conditions (i.e., \( u_n(0,t) = T_0, \ u_n(l,t) = T_1, \ 0 \leq t < \infty \)), the series \( \sum_{n=1}^{\infty} c_n u_n(x,t) \) will generally not satisfy them.

The approach we will use is based upon what to expect as \( t \rightarrow \infty \). Since the heat equation tends to flatten out the temperature profile as \( t \rightarrow \infty \), we expect the temperature to approach a linear profile interpolating the boundary temperatures; i.e., we expect:

\[
\lim_{t \to \infty} u(x,t) = \varphi(x) = T_0 + \left( \frac{T_1 - T_0}{l} \right) x
\]

**Note:** \( \varphi(x) \) is an equilibrium, steady state solution of the heat equation since \( \varphi_x(x) = 0 \) and \( \varphi_{xx}(x) = 0 \).

**Change of dependent variable:**

Let \( w(x,t) = u(x,t) - \varphi(x) \). The problem for \( w(x,t) \) becomes one that we have already solved.

**Problem for \( w(x,t) \):**

\[
\begin{align*}
&w_t(x,t) = k w_{xx}(x,t), \quad 0 < x < l, \quad 0 < t < \infty \quad (\text{since } w_t - k w_{xx} = (u_t - k u_{xx}) - (\varphi_t - k \varphi_{xx})) \\
&w(0,t) = W(l,t) = 0, \quad 0 \leq t < \infty \\
&w(0,t) = W(l,t) = 0, \quad 0 \leq t < \infty \quad (\text{since } W(0,t) = u(0,t) - \varphi(0) = T_0 - T_0 = 0 \text{ and } W(l,t) = u(l,t) - \varphi(l) = T_1 - T_1 = 0) \\
&w(x,0) = u(x,0) - \varphi(x) = f(x) - \varphi(x), \quad 0 \leq x \leq l.
\end{align*}
\]
Therefore, the problem for \( w(x,t) \) becomes a zero temperature ends problem.

\[
\begin{align*}
\text{w(x,t) solution:} & \quad w(x,t) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{L} \right) \\
& \quad \text{where } c_n = \frac{2}{L} \int_0^L (f(x) - u(x)) \sin \left( \frac{n\pi x}{L} \right) dx \\
& \quad n=1,2,3,\ldots
\end{align*}
\]

\[
\begin{align*}
\text{U(x,t) solution:} & \quad u(x,t) = \frac{w(x,t) + u(x)}{2} \\
& \quad \text{where } c_n = \frac{2}{L} \int_0^L (f(x) - u(x)) \sin \left( \frac{n\pi x}{L} \right) dx \\
& \quad n=1,2,3,\ldots
\end{align*}
\]

**Note:** Since \( \lim_{t \to \infty} w(x,t) = 0, \quad 0 \leq x \leq L \), we have:

\[
\lim_{t \to \infty} u(x,t) = u(x), \quad 0 \leq x \leq L \quad \text{as anticipated.}
\]

**Example:**

\[
\begin{align*}
\frac{\partial u}{\partial t} & = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad 0 < t < \infty \\
\begin{align*}
u(0,t) & = 50, \quad \nu(2,t) = 150, \quad 0 \leq t < \infty \\
u(x,0) & = \begin{cases} 
50, & 0 \leq x \leq 1 \\
150, & 1 < x \leq 2 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
u(x) & = 50 + \frac{(150-50) x}{2} = 50 + 50 x, \quad 0 \leq x \leq 2
\end{align*}
\]

\[
\begin{align*}
w(x,0) & = f(x) - u(x) = \begin{cases} 
50 x, & 0 \leq x \leq 1 \\
100 - 50 x, & 1 < x \leq 2
\end{cases}
\end{align*}
\]

Therefore:

\[
\begin{align*}
\frac{\partial w}{\partial t} & = k \frac{\partial^2 w}{\partial x^2}, \quad 0 < x < 2, \quad 0 < t < \infty \\
w(0,t) & = 0, \quad 0 \leq t < \infty \\
w(x,0) & = \begin{cases} 
50 x, & 0 \leq x \leq 1 \\
100 - 50 x, & 1 < x \leq 2
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\therefore \quad w(x,t) & = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{2} \right) \\
& \quad \text{where: } c_n = \frac{2}{L} \left[ \int_0^1 50 x \sin \left( \frac{n\pi x}{2} \right) dx + \int_1^2 (100-50 x) \sin \left( \frac{n\pi x}{2} \right) dx \right] \\
& \quad n=1,2,\ldots
\end{align*}
\]

and: \( u(x,t) = w(x,t) + 50 + 50 x \)

The solution is obtained and plotted (with \( k=0.1 \)) using Mathematica on next pages.
Problem:
\[ u(x, t) = xu(x, t), \quad 0 < x < 2, \quad 0 < t < \infty \]
\[ u(0, t) = 50, \quad u(2, t) = 150, \quad 0 < t < \infty \]
\[ u(x, 0) = f(x) = \begin{cases} 
50 + 100x & \text{if } 0 \leq x \leq 1 \\
150 & \text{if } 1 < x \leq 2 
\end{cases} \]

Steady state limiting temperature profile:
\[ v(x) := 50 + 50 \times x; \]

Initial condition for \( w(x, 0) \):
\[ w(x, 0) = \begin{cases} 
50x & \text{if } 0 \leq x \leq 1 \\
100 - 50x & \text{if } 1 < x \leq 2 
\end{cases} \]

Determination of series coefficients:

\[
\int \left[ 50 \times x \times \sin(n \pi x / 2), \{x, 0, 1\} \right] + \\
\int \left[ (100 - 50 \times x) \times \sin(n \pi x / 2), \{x, 1, 2\} \right]
\]

\[
\frac{50 \left( -2 n \pi \cos\left[\frac{n \pi}{2}\right] + 4 \sin\left[\frac{n \pi}{2}\right] \right)}{n^2 \pi^2} + \frac{100 \left( n \pi \cos\left[\frac{n \pi}{2}\right] + 2 \sin\left[\frac{n \pi}{2}\right] - 2 \sin[n \pi]\right)}{n^2 \pi^2}
\]

Simplify:

\[
\frac{50 \left( -2 n \pi \cos\left[\frac{n \pi}{2}\right] + 4 \sin\left[\frac{n \pi}{2}\right] \right)}{n^2 \pi^2} + \frac{100 \left( n \pi \cos\left[\frac{n \pi}{2}\right] + 2 \sin\left[\frac{n \pi}{2}\right] - 2 \sin[n \pi]\right)}{n^2 \pi^2}
\]

\[
800 \sin\left[\frac{n \pi}{4}\right]^2 \sin\left[\frac{n \pi}{2}\right]
\]

\[
c[n_] := \frac{800 \sin\left[\frac{n \pi}{4}\right]^2 \sin\left[\frac{n \pi}{2}\right]}{n^2 \pi^2};
\]

\[ \text{Kappa} = 0.1 \]
\[ \text{kappa} = 0.1; \]

\[ w[x_, t_, n_] := \sum[c[n] \times \exp\left[\left(-\left(n \times \pi / 2\right)\right) \times \text{kappa} \times t\right] \times \sin[n \times \pi x / 2], \{n, 1, N\}] ; \]

Check:

\[ \text{Plot}[w[x, 0, 100], \{x, 0, 2\}] \]

\[ \text{Show}[\%10, \text{ImageSize} \rightarrow \text{Small}] \]

![Graph of solution]

50
40
30
20
10
0.5
1.0
1.5
2.0
Solution \( u(x, t) \) defined:

\[
u[x_, t_, N_] := w[x, t, N] + v[x];
\]

The solution is plotted at times \( t = 0, 1, \) and 5:

\[
\text{Plot}[[u[x, 0, 100], u[x, 1, 100], u[x, 5, 100]], \{x, 0, 2\}]
\]

As time \( t \) increases, the solution \( u(x, t) \) tends toward the linear steady state profile.
Exercises:

1. Use trigonometric identities to evaluate the series coefficients in the solutions of the following initial-boundary value problems.

   a) \( u(x,t) = k \int_0^x u(x,t) \, dx \), \( 0 < x < l \), \( 0 < t < \infty \)
   
   \( u(x,0) = u_x(x,t) = 0 \), \( 0 < x < l \), \( 0 < t < \infty \)
   
   \( u(x,0) = 4 \sin \left( \frac{\pi x}{l} \right) \cos^2 \left( \frac{\pi x}{l} \right) \), \( 0 \leq x \leq l \)

   b) \( u(x,t) = k \int_0^x u_x(x,t) \, dx \), \( 0 < x < l \), \( 0 < t < \infty \)

   \( u_x(0,t) = u_x(l,t) = 0 \), \( 0 \leq t \leq \infty \)

   \( u(x,0) = 2 \cos^2 \left( \frac{\pi x}{l} \right) + \sin^2 \left( \frac{2\pi x}{l} \right) \), \( 0 \leq x \leq l \)

   c) \( u(x,t) = k \int_0^x u_x(x,t) \, dx \), \( 0 < x < l \), \( 0 < t < \infty \)

   \( u(0,t) = 10 \), \( u(l,t) = 50 \), \( 0 \leq t \leq \infty \)

   \( u(x,0) = 10 + 40x + 8 \sin^3 \left( \frac{\pi x}{l} \right) \), \( 0 \leq x \leq l \).

2. Solve the following initial-boundary value problems.
   Use Mathematica as you wish to determine the series coefficients.
   Plot the solution at times \( t = 0, 1, \) and 4 on the same graph, summing the first 100 terms as a suitable approximation of the solution.

   a) \( u_t(x,t) = 0.1 \int_0^x u_x(x,t) \, dx \), \( 0 < x < 4 \), \( 0 < t < \infty \)

   \( u(x,0) = u_x(4,t) = 0 \), \( 0 \leq t \leq \infty \)

   \( u(x,0) = f(x) = \begin{cases} 4, & 2 \leq x \leq 3 \\ 0, & 0 \leq x < 2 \text{ and } 3 < x \leq 4 \end{cases} \)

   b) \( u_t(x,t) = 0.1 \int_x^l u_x(x,t) \, dx \), \( 0 < x < 4 \), \( 0 < t < \infty \)

   \( u_x(0,t) = u_x(4,t) = 0 \), \( 0 \leq t \leq \infty \)

   \( u(x,0) = f(x) = \begin{cases} 4, & 2 \leq x \leq 3 \\ 0, & 0 \leq x < 2 \text{ and } 3 < x \leq 4 \end{cases} \)

   c) \( u_t(x,t) = 0.1 \int_x^l u_x(x,t) \, dx \), \( 0 < x < 4 \), \( 0 < t < \infty \)

   \( u(0,t) = 0 \), \( u(4,t) = 50 \), \( 0 \leq t \leq \infty \)

   \( u(x,0) = 50 \sin \left( \frac{\pi x}{4} \right) \), \( 0 \leq x \leq 4 \)
Additional Problems:

1. Heat equation with a source/sink term proportional to temperature:

Problem:

\[ U_t(x,t) = k \cdot U_{xx}(x,t) + \alpha U(x,t), \quad 0 < x < l, \quad 0 < t < \infty \]

\[ u(0,t) = u(l,t) = 0, \quad 0 < t < \infty \]

\[ u(x,0) = f(x), \quad 0 \leq x \leq l \quad \text{(with } f(0) = f(l) = 0) \]

Make the change of dependent variable: \[ U(x,t) = \mathcal{E}^{\alpha t} W(x,t) \]

Substituting:

\[ U_t = \mathcal{E}^{\alpha t} W_t + \mathcal{E}^{\alpha t} W = k \mathcal{E}^{\alpha t} W_{xx} + \alpha \mathcal{E}^{\alpha t} W \]

\[ \mathcal{E}^{\alpha t} w_t = k \mathcal{E}^{\alpha t} W_{xx} \quad \text{or} \quad W(x,t) = k W_{xx}(x,t), \quad 0 < x < l, \quad 0 < t < \infty \]

Also: \[ \mathcal{E}^{\alpha t} w(0,t) = \mathcal{E}^{\alpha t} W(l,t) = 0, \quad 0 < t < \infty \quad \text{or} \quad W(0,t) = W(l,t) = 0, \]

and: \[ u(x,0) = W(x,0) = f(x), \quad 0 \leq x \leq l \]

Summary:

\[ W_t(x,t) = k W_{xx}(x,t), \quad 0 < x < l, \quad 0 < t < \infty \]

\[ W(0,t) = W(l,t) = 0, \quad 0 < t < \infty \]

\[ W(x,0) = f(x), \quad 0 \leq x \leq l \]

This is a zero temperature ends problem. Therefore:

\[ u(x,t) = \mathcal{E}^{\alpha t} W(x,t) = \sum_{n=1}^{\infty} C_n \mathcal{E}^{\alpha t} \sin \left( \frac{\pi n x}{l} \right) \]

\[ C_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{\pi n x}{l} \right) dx, \quad n = 1, 2, \ldots \]

Remark: We could solve the insulated ends version of this problem using the same change of dependent variable. Although we assumed \( \alpha = \text{constant} \), we can also deal with the case where \( \alpha = \alpha(t) \). In that case the appropriate change of dependent variable would be:

\[ U(x,t) = \mathcal{E}^{\alpha t} W(x,t) \]

2. Constant Flux:

Problem:

\[ U_t(x,t) = k U_{xx}(x,t), \quad 0 < x < l, \quad 0 < t < \infty \]

\[ U_x(0,t) = U_x(l,t) = F_0, \quad 0 < t < \infty \]

\[ u(x,0) = f(x), \quad 0 \leq x \leq l \quad \text{(with } f(0) = f(l) = F_0) \]

Make the change of dependent variable: \[ U(x,t) = W(x,t) + F_0 \cdot x \]
The problem for $w(x,t)$ becomes: 
\[ W(x,t) = \int_{0}^{\infty} W_x(x,t), \quad 0 < x < \lambda, \quad 0 < t < \infty \]
\[ W_x(0,t) = W_x(\lambda,t) = 0, \quad 0 < t < \infty \]
\[ W(x,0) = f(x) - F_0 x, \quad 0 < x < \lambda \]

The problem for $w(x,t)$ reduces to the insulated ends problem.
Therefore,
\[ U(x,t) = \frac{Q_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\lambda}\right) + F_0 x, \quad \text{where:} \]
\[ C_n = \frac{2}{\lambda} \int_{0}^{\lambda} \left( f(x) - F_0 x \right) \cos\left(\frac{n\pi x}{\lambda}\right) dx, \quad n = 0, 1, 2, 3, \ldots \]

3. Finite Length Bar with one end at zero temperature and the other end insulated:

Problem: 
\[ U_t(x,t) = k U_{xx}(x,t), \quad 0 < x < \lambda, \quad 0 < t < \infty \]
\[ U(0,t) = 0, \quad U'_{x}(\lambda,t) = 0, \quad 0 < t < \infty \]
\[ U(x,0) = f(x), \quad 0 < x < \lambda \quad (\text{with } f(0) = 0, f'(\lambda) = 0) \]

Separation of Variables:
Looking for solutions $u(x,t) = X(x)T(t)$ leads to:
\[ X''(x) + \sigma X(x) = 0, \quad 0 < x < \lambda \]
\[ X(0) = 0, \quad X'_{x}(\lambda) = 0 \]
\[ T'(t) + \sigma T(t) = 0, \quad t > 0 \]

Assuming $\sigma$ real-valued, we consider the 3 possibilities:

1. $\sigma = 0$:
   \[ X''(x) = 0 \Rightarrow X(x) = k_1 x + k_2. \]
   Imposing the boundary conditions:
   \[ X(0) = k_2 = 0, \quad X'(\lambda) = k_1 \lambda = 0. \]
   \[ X(x) = 0 \text{ and no nontrivial solutions if } \sigma = 0. \]

2. $\sigma < 0$:
   With $\sigma = -\lambda^2$, \[ X''(x) + \lambda^2 X(x) = 0 \Rightarrow X(x) = k_1 e^{\lambda x} + k_2 e^{-\lambda x} \]
   Imposing the boundary conditions:
   \[ \begin{bmatrix} 1 & 1 \\ \lambda e^{\lambda \lambda} & -e^{-\lambda \lambda} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
   \[ \det = -\lambda(e^{\lambda \lambda} + e^{-\lambda \lambda}) = -2\lambda \cosh(\lambda) \neq 0. \quad \therefore \lambda = k_1 = 0 \text{ and no nontrivial solutions}, \]

3. $\sigma > 0$:
   With $\sigma = \lambda^2 > 0$:
   \[ X''(x) + \lambda^2 X(x) = 0, \quad 0 < x < \lambda \quad \text{with } X(0) = 0, X'(\lambda) = 0. \]
   \[ X(x) = k_1 \cos(\lambda x) + k_2 \sin(\lambda x) \]
   Imposing the boundary conditions:
   \[ X(0) = k_1 = 0, \quad X(\lambda) = k_2 \sin(\lambda \lambda) \quad \text{and } X'(\lambda) = \lambda \cos(\lambda \lambda) = 0. \]
   Since $\lambda \neq 0$, \[ \cos(\lambda \lambda) = 0 \Rightarrow \lambda_n \lambda = (n - \frac{1}{2})\pi, \quad n = 1, 2, 3, \ldots \]
   \[ X_n(x) = \sin(\frac{(2n-1)\pi}{2}), \quad n = 1, 2, 3, \]
   \[ \therefore C_n = \frac{4}{\lambda} \int_{0}^{\lambda} \left( f(x) - F_0 x \right) \sin\left(\frac{(2n-1)\pi x}{2}\right) dx \]
   \[ \therefore U(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{(2n-1)\pi x}{2}\right) \]
   \[ T_n(t) = e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 k t} \quad \text{and:} \]
   \[ U(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{(2n-1)\pi x}{2}\right) \]
   \[ T_n(t) = e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 k t} \quad \text{and:} \]
Since \( \int_0^l \sin \left( \frac{(2n-1)\pi x}{2l} \right) \sin \left( \frac{(2m-1)\pi x}{2l} \right) dx = \frac{1}{2} \int_0^l \left[ \cos \left( \frac{(m-n)\pi x}{l} \right) - \cos \left( \frac{(m+n-2)\pi x}{l} \right) \right] dx \)

\( \begin{cases} \frac{\delta}{2} & n=m \quad \text{we have:} \\ 0 & n \neq m \end{cases} \)

**Example:**

\( u_t(x,t) = 0 \quad u_x(x,t); \quad 0 < x < 1, \quad 0 < t < \infty \)

\( u(x,0) = 0, \quad u_x(1,t) = 0, \quad 0 < t < \infty \)

\( u(x,0) = 8 \sin^3 \left( \frac{\pi x}{2} \right), \quad 0 < x < 1 \)

Therefore:

\( u(x,t) = \sum_{n=1}^\infty C_n \cos \left( \frac{(2n-1)\pi x}{2} \right) \sin \left( \frac{(2n-1)\pi x}{2} \right) \)

\( u(x,0) = \sum_{n=1}^\infty C_n \sin \left( \frac{(2n-1)\pi x}{2} \right) = 8 \sin^3 \left( \frac{\pi x}{2} \right) = 6 \sin \left( \frac{\pi x}{2} \right) - 2 \sin \left( \frac{3\pi x}{2} \right), \quad 0 < x < 1 \)

\( \therefore C_1 = 6, \quad C_2 = -2, \quad C_n = 0, \quad n \geq 2 \) and:

\( u(x,t) = 6 \sum_{n=1}^\infty \left[ \cos \left( \frac{(2n-1)\pi x}{2} \right) \sin \left( \frac{(2n-1)\pi x}{2} \right) \right] - 2 \sum_{n=1}^\infty \sin \left( \frac{3\pi x}{2} \right) \)

**Mathematica plots of the solution at times:**

\( t = 0, \quad 1 \) and \( 5 \) are shown on the next page.

**Note that** \( \lim_{t \to \infty} u(x,t) = 0, \quad 0 < x < l \) **for this problem since thermal energy leaks out the end kept at zero temperature.**

**Fourier Series:**

**Def.:** A **Fourier Series** is a series of the form:

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(n\Theta \right) + b_n \sin\left(n\Theta \right) \right) \]

We shall be interested in the case where \( \Theta \to \frac{\pi x}{l}, \) i.e.

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(n\pi x / l \right) + b_n \sin\left(n\pi x / l \right) \right) \]

**Our motivation is two-fold.**

(i) Note that if \( a_n = 0, \quad n = 1; 2, \ldots, \) the series reduces to that representing the initial condition in the zero temperature ends problem. If \( b_n = 0, \quad n = 1, 2, \ldots, \) the series reduces to that representing the initial condition in the insulated ends problem. The theory developed for Fourier Series will provide justification for our calculations and explain how the Series converge to the initial temperature profile.

(ii) Fourier Series is important in itself. It provides a spectral decomposition of periodic signals, telling us what frequencies (harmonics) are present and with what strengths,
Problem:
\[ u_t(x,t) = 0.1u_{xx}(x,t), \quad 0 < x < 1, \quad 0 < t < \infty \]
\[ u(0,t) = 0, \quad u_x(1,t) = 0, \quad 0 < t < \infty \]
\[ u(x,0) = 8\sin^3\left(\frac{x\pi}{2}\right), \quad 0 \leq x \leq 1 \]

Solution: \[ u(x,t) = 6e^{-\left(\frac{5}{2} + 0.1t\right)\sin\left(\frac{x\pi}{2}\right)} - 2e^{-\left(\frac{25}{2} + 0.1t\right)\sin\left(\frac{3x\pi}{2}\right)} \]

Plot of the solution at times \( t = 0, 1 \) and 5. Thermal energy leaks out of the left end of the bar and the solution approaches zero as time increases.

\[
\begin{align*}
  u[x_-, t_] &:= 6 \cdot \text{Exp}[-((\text{Pi}/2)^2) \cdot 0.1 \cdot t] \cdot \text{Sin}[\text{Pi} \cdot x/2] - \\
  &\quad 2 \cdot \text{Exp}[-((3 \cdot \text{Pi}/2)^2) \cdot 0.1 \cdot t] \cdot \text{Sin}[3 \cdot \text{Pi} \cdot x/2]; \\
\text{Plot}[\{u[x, 0], u[x, 1], u[x, 5]\}, \{x, 0, 1\}] &
\end{align*}
\]
Def: \( f: (-\infty, \infty) \to \mathbb{R} \) is a periodic function having fundamental period \( T \) if:
\[
f(x + nT) = f(x), \quad 0 \leq x \leq T, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots
\]

Consider the collection of functions \( \{ 1, \sin(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \}_{n=1}^{\infty} \).

**Orthogonality Relations:**

One can show that:

(i) \[
\int_{-\frac{L}{2}}^{\frac{L}{2}} 1 \, dx = L
\]

(ii) \[
\int_{-\frac{L}{2}}^{\frac{L}{2}} 1 \cdot \cos\left(\frac{n\pi x}{L}\right) \, dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} 1 \cdot \sin\left(\frac{n\pi x}{L}\right) \, dx = 0, \quad n = 1, 2, 3, \ldots
\]

(iii) \[
\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \begin{cases} L, & n = m \pm 1 \\ 0, & n \neq m \end{cases}
\]

(iv) \[
\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \, dx = \begin{cases} L, & n = m \pm 1 \\ 0, & n \neq m \end{cases}
\]

(v) \[
\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \, dx = 0 \quad \text{for all} \quad m, n.
\]

Therefore, the set of functions \( \{ 1, \sin(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \}_{n=1}^{\infty} \) considered as defined on \(-\frac{L}{2} \leq x \leq \frac{L}{2}\), form a mutually perpendicular set of "vectors" (not of unit length.)

**Remark:** In the finite length bar problems, the physical extent of the bar is \(-\frac{L}{2} \leq x \leq \frac{L}{2}\) and the integrals encountered were of the form \( \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \, dx \). In the Fourier Series discussion, we are now dealing with integrals \( \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \, dx \). This difference will have to be reconciled.

**Coefficient Formulas:** Let \( f: (-\infty, \infty) \to \mathbb{R} \) be periodic with period \( T = 2L \). Let:
\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L})) + b_n \sin(\frac{n\pi x}{L}).
\]

Then,
\[
a_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos(\frac{n\pi x}{L}) \, dx, \quad n = 0, 1, 2, 3, \ldots
\]
\[
b_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin(\frac{n\pi x}{L}) \, dx, \quad n = 1, 2, 3, \ldots
\]

We obtain these formulas in a manner similar to those used for determining the \( \{ a_n \} \) coefficients in the heat conduction problems.
We will:
(i) consider some special cases,
(ii) evaluate the series coefficients for some examples, and
(iii) show how Fourier Series theory and calculations relate to the
heat conduction problems.

Special Cases:
Let \( f(x) \) be defined on \(-L \leq x \leq L\) or \(-\infty < x < \infty\). Recall that:
(i) \( f(x) \) is an even function if \( f(-x) = f(x) \).
(ii) \( f(x) \) is an odd function if \( f(-x) = -f(x) \).

Examples:
(i) \( \cos \left( \frac{\pi x}{L} \right) \) is an even function. (\( \cos(-\theta) = \cos(\theta) \).)
(ii) \( \sin \left( \frac{\pi x}{L} \right) \) is an odd function. (\( \sin(-\theta) = -\sin(\theta) \).)

Some relations:
For brevity let \( e \) denote an even function and \( o \) an odd function,
Then:
(i) \( e \cdot e = e \), \( e \cdot o = 0 \), \( o \cdot o = o \)
(ii) \( e' = e \), \( e'_o = o \), \( o'_o = e \)
(iii) If \( e(x) \) is an even function:
\[
\int_{-L}^{L} e(x) dx = 2 \int_{0}^{L} e(x) dx \quad \text{since} \quad \int_{-L}^{L} e(x) dx = \int_{0}^{L} e(x) dx
\]
(iv) If \( o(x) \) is an odd function:
\[
\int_{-L}^{L} o(x) dx = 0 \quad \text{since} \quad \int_{0}^{L} o(x) dx = -\int_{0}^{L} o(x) dx
\]

Relevance to Fourier Series:
(i) If \( f(x) \) is an even function, then:
(i) \( f(x) \cos \left( \frac{n \pi x}{L} \right) \) is even (\( e \cdot e = e \))
(ii) \( f(x) \sin \left( \frac{n \pi x}{L} \right) \) is odd (\( e \cdot o = 0 \))

Therefore:
\[
A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) dx = 2 \frac{1}{L} \int_{0}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) dx
\]
\[
B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) dx = 0
\]
and:
\[
f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n \pi x}{L} \right) \quad \text{a Fourier Cosine Series}
\]
with \( A_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) dx \), \( n = 0, 1, 2, \ldots \).
b) If \( f(x) \) is an odd function, then:

(i) \( f(x) \cos \left( \frac{nm\pi x}{\lambda} \right) \) is odd (0.e = 0)

(ii) \( f(x) \sin \left( \frac{nm\pi x}{\lambda} \right) \) is even (0.e = e)

Therefore:

\[
O_n = \frac{1}{\lambda} \int_{-\lambda}^{\lambda} f(x) \cos \left( \frac{nm\pi x}{\lambda} \right) dx = 0, \quad n=1,2,3,\ldots
\]

\[
b_n = \frac{2}{\lambda} \int_{0}^{\lambda} f(x) \sin \left( \frac{nm\pi x}{\lambda} \right) dx = 2 \int_{0}^{\lambda} f(x) \sin \left( \frac{nm\pi x}{\lambda} \right) dx, \quad n=1,2,3,\ldots
\]

and

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{nm\pi x}{\lambda} \right), \quad \text{a Fourier Sine Series}
\]

with:

\[
b_n = \frac{2}{\lambda} \int_{0}^{\lambda} f(x) \sin \left( \frac{nm\pi x}{\lambda} \right) dx, \quad n=1,2,3,\ldots
\]

These special cases lead to series representations and coefficient formulas identical to those occurring in the heat equation problems. We will consider this further after studying two examples.

**Example (square wave):** \( f: (-\infty, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} 
2, & 0 \leq x \leq 1 \\
0, & 1 < x < 2 \\
f(x+2n) = f(x), & n=\pm1, \pm2, \ldots
\end{cases} \)

\[T=2\lambda = 2\]

\( f(x) \) by itself is neither even nor odd. The function \( f(x-1) \), however, is essentially odd. (The definition fails only at isolated points, e.g., \( x=0 \). This does not affect the coefficient integrals.)

Therefore, we expect \( f(x-1) = \sum_{n=1}^{\infty} b_n \sin (nm\pi x) \quad (\lambda=1) \)

or:

\[
f(x) = 1 + \sum_{n=1}^{\infty} b_n \sin (nm\pi x) \quad (\text{i.e., } a_0=1, \quad a_n=0, \quad n \geq 1)
\]

As a check:

\[
a_0 = \frac{1}{\lambda} \left[ \int_{-\lambda}^{0} dx + \int_{0}^{\lambda} 2 dx \right] = 2 \quad \text{so} \quad a_{\pi/2} = 1
\]

\[
a_n = \frac{1}{\lambda} \left[ \int_{-\lambda}^{0} dx + \int_{0}^{\lambda} 2 \cos (nm\pi x) dx \right] = \frac{2}{n\pi} \sin (n\pi x) \bigg|_{0}^{1} = 0, \quad n \geq 1.
\]